Modeling and Control of Multiple-Contact Manipulation without Modeling Friction

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***Abstract*— Multiple contact manipulation often involves contacts that slip or lose contact over time. This slipping introduces frictional reaction forces between the manipulator and the surface. However, in complex systems in uncertain environments, these frictional interactions may not be well-characterized. This paper discusses the reduction of such external forces in multiple contact systems. We then present two examples: a macro-scale multi-point manipulator and MEMS scratch drive actuators (SDA) array to illustrate how the techniques presented can simplify the control design for such devices.**

I. Introduction

Many mechanical systems, though intrinsically second order in their governing dynamics, can be adequately described by first order equations of motion. That is, one can often propose a "kinematic" version of the governing equations of motion for the purposes of system analysis or control design. The benefits of this simplification are numerous: the dimension of the state space drops by half, the control inputs go from being force inputs to being velocity inputs (which are often more easily realized in practice), and the governing equations typically take a simpler form than the full dynamic model. Additionally, kinematic systems, although potentially nonlinear, do not typically involve drift terms. There is a greater quality and quantity of nonlinear control results available for driftless systems, as compared to systems with drift. See [9], [10], [19] for just a few examples.

Recently there has been quite a bit of attention given to the advantages of kinematic systems. Intuitively, we say that a system is kinematic whenever solutions to the kinematic equations are in direct correspondence to solutions to the dynamic equations. In particular, several groups have studied these advantages formally from the perspective of relating solutions to kinematic equations to solutions of dynamic equations, including work by Bullo and Lynch [7], and Lewis et al. [5], [4].

This paper is being written partially in response to reviews from our contribution to last year's ACC [13], where a control strategy was used for multi-point manipulation that uses a hybrid estimator combined with a kinematic reduction to design a controller that does depends minimally on frictional interface dynamics. All reviewers of that work found this surprising, leading to the present, careful presentation of how such a reduction works.

The main result of this paper can be stated roughly as follows. In order to guarantee that a mechanical system with external disturbance forces be reduced to a kinematic one without disturbance forces, it is sufficient that the mechanical system be kinematic and the input vector fields corresponding to the external disturbance forces be spanned by the input vector fields corresponding to the external control inputs. Despite the simplicity of this result, it has a substantial advantage from the perspective of control design. In particular, it allows one to avoid writing down specific models for friction, which is crucial in micro-scale physics (such as MEMS manipulation) and other systems where the external environment is not well characterized. It is this unintuitive use of a reduction that we wish to pursue. We use several examples to illustrate how these kinematic reductions can be used to encode the effects of frictional interactions in a hybrid, kinematic structure that only includes the stick/slip state effects of friction rather than explicitly modeling friction.

The paper is organized as follows. Section II gives a short description of a motivating example. Section III gives the background on the geometry of kinematic reductions. Section IV and Section V discuss the main results of the paper, which essentially states that if a system is kinematic and the disturbance vector fields are spanned by the control vector fields, then the system is reducible to one without disturbances. These results are then applied to intermittent contact systems, and it is shown in Section VII that these systems admit a description that completely excludes the reaction forces due to frictional sliding, replacing them with a hybrid description that only encodes the stick/slip state of the interaction. In Section VIII we then discuss a specific example of MEMS manipulation and an example of multiple contact manipulation and how these modeling techniques can be useful in those contexts.

II. Motivation: Multiple-Contact Manipulation

Consider the mechanism in Fig.1. It has eight degrees of freedom, all independently actuated by a DC brushless motor. The motion of the tips of the "fingers" can be constrained to be in a horizontal plane, so it can be used as a manipulation surface. However, the force any given finger exerts is constrained on a line–no finger can exert any "side-ways" force. In such cases friction forces and intermittent contact play an important role in the overall system dynamics, leading to non-smooth dynamical system

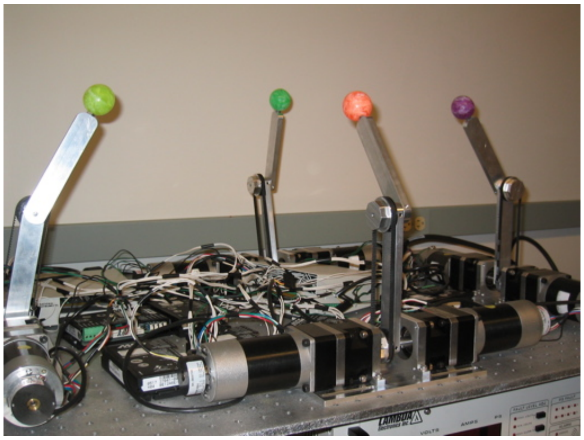

Fig. 1. An 8 degree-of-freedom four “finger” manipulator that manipulates objects it supports. The fingers are constrained, so stick/slip transitions between the actuator end-effectors and the manipulated object must occur when the actuators move.

behavior. The question is how to control the position and orientation of a supported object without being sensitive to the details of how the frictional stick/slip interactions complicate the dynamics.

III. Background: Kinematic Reduction

In discussing kinematic reductions, we follow [5], [6]. A *simple mechanical control system* consists of a manifold Q of dimension n, a Riemannian metric G that defines the kinetic energy, a set of constraints represented as a constraint distribution D, and a set of external forces. Associated with a Riemannian metric G are what are called *Christoffel symbols.*

Definition 3.1: The *Christoffel symbols* associated with the metric G are

$$\Gamma^i_{jk} = \frac{1}{2} G^{il} \left(\frac{\partial G_{jl}}{\partial q^k} + \frac{\partial G_{kl}}{\partial q^j} - \frac{\partial G_{jk}}{\partial q^l} \right) \tag{1}$$

where summation over repeated indices is implied used unless otherwise stated, and upper indices indicate the inverse. Also associated with the Riemannian metric is the *affine connection*, which assigns to a pair of vector fields X and Y another vector field $\nabla_X Y$. This is referred to as the *covariant derivative* of Y with respect to X.

Definition 3.2: In coordinates, the covariant derivative of Y with respect to X is

$$^G\nabla_X Y = \left(\frac{\partial Y^i}{\partial q^j} X^j + \Gamma^i_{jk} X^j Y^k \right) \frac{\partial}{\partial q^j} \tag{2}$$

With this, the Euler-Lagrange equations can be written as

$$^G\nabla_{c'(t)} c'(t) = u^a(t) Y_a(c(t)) \tag{3}$$

where $t \mapsto c(t)$ is a path on Q and $c'(t) = \frac{d}{dt} c(t)$ [5], [6]. In this equation $^G\nabla$ is the constrained affine connection associated with the Riemannian metric (kinetic energy) G and Y_a are force vector fields associated with forces u^a.

We know that Eq. (3) is a second order differential equation evolving on the manifold Q. On the other hand, given input velocities $\overline{u}^\alpha$, *kinematic* equations can be written in the form:

$$\dot{q}(t) = \overline{u}^\alpha(t) X_\alpha(q(t)) \tag{4}$$

Our goal in this section is to formally reduce Eq. (3) to Eq. (4). If $\{X_i\}$ are kinematic vector fields and $\{Y_j\}$ are dynamic vector fields, we let the *distributions* D_{kin} and D_{dyn} be defined by $D_{kin} = span\{X_i\}$ and $D_{dyn} = span\{Y_j\}$. A solution to a control system is defined as follows.

Definition 3.3: Let Σ_s be a smooth control system $\dot{q} = f(q,u)$ on a smooth manifold M and let $u \in U \subseteq \mathbb{R}^m$. A $(\mathcal{U},\mathcal{T})$*-solution* to Σ_s is a pair (c,u), where $u : [0,T] \to U$ and $c : [0,T] \to M$ satisfy $c'(t) = f(c(t), u(t))$.

Note that Def. 3.3 only makes sense for first order equations evolving on M and Eq. (3) is a second order differential equation evolving on Q. Hence, we must rewrite Eq. (3) as a first order equation evolving on TQ. To do this, we must use the *vertical lift*, defined by

$$verlift(X)(v_q) = \frac{d}{dt}|_{t=0} v_q + tX(q),$$

and the *geodesic spray*, defined in coordinates by

$$Z = v^i \frac{\partial}{\partial q^i} - \Gamma^i_{jk} v^j v^k \frac{\partial}{\partial v^i}.$$

Now let τ_Q

$$\tau_Q : \begin{array}{ccc} TQ & \to & Q \\ v_q & \mapsto & q \end{array}$$

denote the tangent bundle projection. Then, Eq. (3) written as a first order system evolving on TQ is

$$\dot{v}(t) = Z(v(t)) + u^a(t) verlift(Y_a(\tau_Q \circ v(t))) \tag{5}$$

where $v(t) \in TQ$. We now can define what it means for a mechanical system of the form in Eq. (3) to be $(\mathcal{U},\overline{\mathcal{U}})$ reducible to Eq. (4).

Definition 3.4: Let ∇ be an affine connection on Q, and let $\mathcal{U}$ and $\overline{\mathcal{U}}$ be two families of control functions. The system in Eq. (3) is $(\mathcal{U},\overline{\mathcal{U}})$*-reducible* to the system in Eq. (4) if the following two conditions hold:

i) for each $(\mathcal{U},\mathcal{T})$-solution (η,u) of the dynamic Eq. (3) with initial conditions $\eta(0)$ in the distribution D_{kin}, there exists a $(\overline{\mathcal{U}},\mathcal{T})$-solution $(\gamma,\overline{u})$ of the kinematic Eq. (4) with the property that $\gamma = \tau_Q \circ \eta$;

ii) for each $(\overline{\mathcal{U}},\mathcal{T})$-solution $(\gamma,\overline{u})$ of the kinematic Eq. (4), there exists a $(\mathcal{U},\mathcal{T})$-solution (η,u) of the dynamic Eq. (3) with the property that $\eta(t) = \gamma'(t)$ for almost every $t \in [0,T]$.

Condition i) says that for every solution of a dynamic system there must exist a kinematic solution that is the projection of the dynamic system. In the case of a vehicle, this corresponds to requiring that for every *trajectory* of the vehicle there exists a corresponding *path* that can be obtained from kinematic considerations alone. Condition ii) says that for every kinematic solution there must exist a dynamic solution that is equal to the kinematic solution coupled with its time derivative (so that it lies in TQ). This means that there must exist a dynamic solution for every feasible kinematic path.

Let $\chi^{\infty}(D)$ denote those C^{∞} vector fields taking values in a distribution D. The following theorem states the local test for Eq. (3) to be $(\mathcal{U},\overline{\mathcal{U}})$ reducible to Eq. (4).

Theorem 3.1 (Lewis [6]): Let ∇ be an affine connection, and let $Y_1,\ldots,Y_m$ and $X_1,\ldots,X_{\overline{m}}$ be vector fields on a manifold Q. The control system in Eq. (3) is $\left(\mathcal{U},\overline{\mathcal{U}}\right)$ *reducible* to a system of the form in Eq. (4) if and only if the following two conditions hold:

i) $span_{\mathbb{R}}\{X_1(q),\ldots,X_m(q)\} = span_{\mathbb{R}}\{Y_1(q),\ldots,Y_{\overline{m}}(q)\}$ for each $q \in Q$ (in particular, $\overline{m}=m$)

ii) $\langle X : Y\rangle \in \chi^{\infty}(D_{dyn})$ for every $X,Y \in \chi^{\infty}(D_{dyn})$ where $\langle\cdot,\cdot\rangle$ is the symmetric product of vector fields defined by

$$\langle X : Y\rangle = {}^{G}\nabla_X Y + {}^{G}\nabla_Y X \qquad (6)$$

This theorem says that if the input distributions of both the kinematic system and the dynamic system are the same and the dynamic system is closed under symmetric products, then the system is kinematic. The following Lemma (from [16]) is a particularly useful result that we will use repeatedly in our analysis of the examples.

Lemma 3.2: Given a "constraint distribution" $D_{con} \subseteq TQ$ which annihilates the constraints $\{\omega_j\}$ and an input distribution D_{dyn}, if $D_{dyn} = D_{con}$ the mechanical system described by $\nabla_{\dot{q}}\dot{q} = u^{\alpha}Y_{\alpha}$ is $(\mathcal{U},\overline{\mathcal{U}})$ reducible.
This says that whenever when the constraint distribution is covered by the input distribution, $(\mathcal{U},\overline{\mathcal{U}})$ reducibility of a multiple model mechanical system is guaranteed regardless of the metric G.

A. Kinematic Reductions for Multiple Model Systems

In order to talk about whether an intermittent contact system is kinematic, we must formalize what we mean mathematically by a dynamic system that changes its dynamics discretely. To do so, we use the formalism of multiple model systems.

Definition 3.5: A control system Σ evolving on a smooth n-dimensional manifold, M, is said to be a ***multiple model driftless affine system (MMDA)*** if it can be expressed in the form

$$\Sigma : \quad \dot{q} = f_1(q,t)u_1 + f_2(q,t)u_2 + \cdots + f_m(q,t)u_m \quad (7)$$

where $q \in M$. For any q and t, the vector field f_i assumes a value in a finite set of vector fields: $f_i \in \{g_{\alpha_i} | \alpha_i \in I_i\}$, with I_i an index set. The vector fields g_{α_i} are assumed to be analytic in (q,t) for all α_i, and the controls $u_i \in \mathbb{R}$ are piecewise constant for all i. Moreover, letting σ_i denote the "switching signals" associated with f_i

$$\begin{array}{rccc} \sigma_i : & M\times\mathbb{R} & \rightarrow & \mathbb{N} \\ & (q,t) & \mapsto & \alpha_i \end{array}$$

the σ_i are measurable in (q,t).

Definition 3.5 implies that the control vector fields may change, or switch, among a finite collection of vector fields, each representing a single smooth model in a set of models $\mathcal{P}$. An example of such a system is a vehicle whose wheels can potentially skid. The system's governing dynamics will vary when the wheels slip or do not slip. We should emphasize that the switching is induced by environmental factors, such as variations in the contact state between rigid bodies. Since the phenomena which govern the switching behavior may not be precisely characterized, we make no assumptions about the nature of the switching functions, except that they are measurable.

To distinguish between the overall control system and the smooth control systems that comprise it, we define the *individual control systems* to be the smooth control systems making up the multiple model system, comprising of $\dot{q} = g_1(q,t)u_1 + \cdots + g_k(q,t)u_k \cdots + g_n(q,t)u_n$ for $g_k(q,t) = g_{\alpha_i}(q,t)$ for some α_i. A system will be termed a ***multiple model affine*** system if it has the form $\dot{q} = f_0(q,t)+f_1(q,t)u_1+f_2(q,t)u_2+\cdots+f_m(q,t)u_m$, where the vector field $f_0(q,t)$ (or "drift term") is also selected from a set of analytic vector fields $\{g_{\sigma_0}(q,t)\}$. We can now state the condition for a multiple model system to be $(\mathcal{U},\overline{\mathcal{U}})$ reducible.

Theorem 3.3: A multiple model system Σ where the individual model components $\Sigma_{\sigma_i,\cdots,\sigma_j}$ are of the form in Eq. (3) (or equivalently the first order form in Eq. (5)) is $(\mathcal{U},\overline{\mathcal{U}})$ reducible iff the individual dynamical models $\Sigma_{\sigma_i,\cdots,\sigma_j}$ are all $(\mathcal{U},\overline{\mathcal{U}})$ reducible.

This result will be important later because it allows us to only be concerned with the kinematic reducibility of a device in a particular contact state, rather than being concerned about the effects of switching between dynamic models.

IV. A Sufficient Condition for Reduction of Bounded External Forces

First we discuss the reduction for single model systems. Therefore, we are currently interested in understanding systems of the form

$${}^{G}\nabla_{c'(t)}c'(t) \in u^a(t)Y_a(c(t)) + d^b(t)V_b(c(t)) \qquad (8)$$

where ${}^{G}\nabla$ is the (possibly constrained) affine connection associated with the Riemannian metric G, d^b is a set of forces corresponding to external disturbances, V_b is the set of corresponding vector fields, u^a is a set of forces corresponding to control inputs, and Y_a are the associated vector fields. Since we are motivated by not wanting to be forced to rely on the correctness of one particular disturbance force model (such as friction force modeling where there are many possible choices of model), we allow, for each index b, the term d^bV_b to be set-valued. If d^bV_b as a set is not convex, then we replace it by its convex hull $co\{d^bV_b\}$ so as to guarantee solutions exist in the Filippov sense [8].

Now, given a system with set-valued disturbances such as in Eq.(8), we ask under what circumstances it can be reduced to a system of the form in Eq. (3). That is, when can we find an equivalent system that does not include external disturbance forces. To make such an equivalence more rigorous, we introduce some definitions, following Section III for guidance.

Definition 4.1: Let Σ_s be a smooth control system $\dot{q} = f(q, u, d)$ on a smooth manifold M and let $u \in U_{\text{input}} \subseteq \mathbb{R}^m$. A $(\mathcal{U}, \mathcal{D}, \mathcal{T})$-*solution* to Σ_s is a triple (c, u, d), where $u : [0, T] \to U$, $d : [0, T] \to U_{\text{disturbance}} \subseteq \mathbb{R}^l$, and $c : [0, T] \to M$ satisfy $c'(t) = f(c(t), u(t), d(t))$.

Again using the definitions found in Section III for guidance, we define the following notion of reduction.

Definition 4.2: Let ∇ be an affine connection on Q, and let $\mathcal{U}$ be a family of control functions and $\mathcal{D}$ be a family of disturbance functions. The system in Eq. (8) is $(\mathcal{U}, \mathcal{D})$-*reducible* to the system in Eq. (3) if for each $(\mathcal{U}, \mathcal{D}, \mathcal{T})$-solution (η_1, u_1, d) of the Eq. (8) there exists a $(\mathcal{U}, \mathcal{T})$-solution (η_2, u_2) of Eq. (3) with $\eta_1(t) = \eta_2(t)$;

Lastly, we would like to be rigorous about what it means for a mechanical system with set-valued disturbances to be reducible to a kinematic system, which leads to the following definition (based on Def. 3.4).

Definition 4.3: Let ∇ be an affine connection on Q, and let $\mathcal{U}$ and $\overline{\mathcal{U}}$ be two families of control functions. The system in Eq. (8) is $\big((\mathcal{U}, \mathcal{D}), \overline{\mathcal{U}}\big)$-*reducible* to the system in Eq. (4) if the following two conditions hold:

i) for each $(\mathcal{U}, \mathcal{D}, \mathcal{T})$-solution (η, u) of the dynamic Eq. (3) with initial conditions $\eta(0)$ in the distribution D_{kin}, there exists a $\big(\overline{\mathcal{U}}, \mathcal{T}\big)$-solution $(\gamma, \overline{u})$ of the kinematic Eq. (4) with the property that $\gamma = \tau_Q \circ \eta$;

ii) for each $\big(\overline{\mathcal{U}}, \mathcal{T}\big)$-solution $(\gamma, \overline{u})$ of the kinematic Eq. (4), there exists a $(\mathcal{U}, \mathcal{D}, \mathcal{T})$-solution (η, u, d) of the dynamic Eq. (3) with the property that $\eta(t) = \gamma'(t)$ for almost every $t \in [0, T]$.

With these definitions, we can state sufficient conditions for $(\mathcal{U}, \mathcal{D})$-reducibility and for $\big((\mathcal{U}, \mathcal{D}), \overline{\mathcal{U}}\big)$-reducibility. Intuitively, this corresponds to being able to guarantee that any solutions that include disturbances can be mapped directly to a solution that has no disturbances.

Lemma 4.1: Assume we have a mechanical system of the form in Eq. (8) with unbounded inputs. Then the system in Eq. (8) is $(\mathcal{U}, \mathcal{D})$-reducible to the mechanical system in Eq. (3) iff this system satisfies $co\{d^b V_b\} \in \text{span}\{Y_a\}$ for all b.

Proof: The condition is clearly necessary because if $co\{d^b V_b\} \notin \text{span}\{Y_a\}$ we automatically have a trajectory that the system cannot follow just using the control inputs. Sufficiency is nearly as clear. Because $co\{d^b V_b\} \in \text{span}\{Y_a\}$ for all b, we know that any selection of $co\{d^b V_b\}$ in the differential inclusion in Eq.(8) can be rewritten as a linear combination of Y_a. That is, $d^b V_b = \sum \beta u^a Y_a$. Since u are unbounded, this always represents an admissible u. Hence, any (η_1, u_1, d) triple can be rewritten as $(\eta_1, u_1 + \sum \beta u^a)$, which is a (U, T) solution of Eq. (3). ■

This means that all trajectories can be planned as if there are no forces due to the terms $d^b V_b$. However, from a control perspective, we are implicitly assuming that some lower-level controller is compensating for the differences between the trajectories of the two systems. Hence, this is effectively a "backstepping" technique.

The next result will get us closer to being able to describe systems with disturbances as kinematic systems. It states that we can effectively decouple the question of $\big(\mathcal{U}, \overline{\mathcal{U}}\big)$-reducibility from that of $(\mathcal{U}, \mathcal{D})$-reducibility.

Lemma 4.2: Suppose that we have a simple mechanical system of the form in Eq. (8). Suppose it is (U, D) reducible and that its reduction satisfies the conditions for $\big(\mathcal{U}, \overline{\mathcal{U}}\big)$-reducibility. Then Eq. (8) is $\big((\mathcal{U}, \mathcal{D}), \overline{\mathcal{U}}\big)$-reducible to a system of the form in Eq. (4).

Proof: We know that the system is (U, D) reducible. This implies that for any solution of Eq. (8) (η_1, u_1, d) there exists a solution of Eq. (3) (η_2, u_2) such that $\eta_1(t) = \eta_2(t)$. Because Eq. (3) is $\big(\mathcal{U}, \overline{\mathcal{U}}\big)$-reducible, we know there exists a solution of Eq. (4) $(\gamma, \overline{u})$ with $\gamma(t) = \tau_Q \circ \eta_2$. For a kinematic solution to Eq. (4) $(\gamma, \overline{u})$, we know that there exists a solution (η_2, u_2) of Eq. (3) with $\eta_2(t) = \gamma\prime(t)$ for almost all $t \in [0, T]$. Lastly, (η_2, u_2) has a corresponding solution to Eq. (8) (η_1, u_1, d). Therefore, Eq. (8) is $\big((\mathcal{U}, \mathcal{D}), \overline{\mathcal{U}}\big)$-reducible to Eq. (4). ■

Note that this could lead to kinematic equations with drift, if one chooses to interpret one of the inputs $\overline{u}^\alpha$ for the kinematic system as a drift term. However, for our purposes it will be appropriate to always view the inputs $\overline{u}^\alpha$ as control inputs.

V. $\big((\mathcal{U}, \mathcal{D}), \overline{\mathcal{U}}\big)$-REDUCIBILITY FOR MULTIPLE MODEL SYSTEMS

We are now interested in finding out when a multiple model of the form

$$
{}^G_\sigma\nabla_{c'(t)} c'(t) \in u^a(t) Y_a^\sigma(c(t)) + d^b(t) V_b^\sigma(c(t)) \qquad (9)
$$

is reducible to a system of the form in Eq.(7).

Theorem 5.1: Let $\sum$ be a multiple model control system where each model is of the form in Eq. (8). Assume all the individual models are $\big((\mathcal{U}, \mathcal{D}), \overline{\mathcal{U}}\big)$-reducible. Then the multiple model system $\sum$ is $\big((\mathcal{U}, \mathcal{D}), \overline{\mathcal{U}}\big)$-reducible

Proof: We need to know if for every triple (c, u, d) solution of Eq.(8) there exists a $(\gamma, \overline{u})$ solution of Eq.(7). Since the individual models are each $\big((\mathcal{U}, \mathcal{D}), \overline{\mathcal{U}}\big)$-reducible, we know that for each individual model solution there exists a kinematic solution. By Theorem 3.3, we know that since the individual models are $\big(\mathcal{U}, \overline{\mathcal{U}}\big)$-reducible the multiple model system is $\big(\mathcal{U}, \overline{\mathcal{U}}\big)$-reducible. This in turn implies the existence of the solutions we require. ■

Notice that this does *not* mean that if every model is $(\mathcal{U}, \mathcal{D})$-reducible that the multiple model system is $(\mathcal{U}, \mathcal{D})$-reducible. In particular, although it is true that $co\{d^b V_b^\sigma\} \subseteq span\{Y_a^\sigma\} \ \forall \ \sigma$ implies

$$
co_\sigma\{co\{d^b V_b^\sigma\}\} \subseteq co_\sigma\{span\{Y_a^\sigma\}\},
$$

this fact does not help us. The term on the left hand side of the equality represents the combined uncertainty arising from both switching and external force modeling. The right hand side of the equality represents the control action and external switching. Hence, solving for the left hand side in terms of input forces would involve being able to dictate the switching signal σ. Hence, we are only able to guarantee

$\big((\mathcal{U},\mathcal{D}),\overline{\mathcal{U}}\big)$-reducibility by using a strong condition, such as

$$co_\sigma\{co\{d^bV_b^\sigma\}\} \subseteq \bigcap_\sigma \{span\{Y_a^\sigma\}\}.$$

Such a strong condition rarely holds for mechanical systems. Because of this technical difficulty, Theorem 5.1 implies that kinematic systems have substantial advantages over dynamic ones when it comes to external force reduction. The main point is that when each of the individual models can be reduced to a kinematic model, so can the multiple model system.

VI. Kinematic Reductions in Closed Loop

Everything discusses so far has implicitly relied on the control being "open-loop." However, we point out here that if one is using a discrete time controller (with one's favorite continuation algorithm, such as zero-order holds) the control is open loop in between controller updates. It was already shown in [16] that $\big(\mathcal{U},\overline{\mathcal{U}}\big)$-reductions are not affected by occasional discontinuities. By the exact same logic, the systems considered here are reducible in discrete time closed-loop if they satisfy the requirements to be reducible under the tests of Lemma 4.2

Lemma 6.1: A discrete-time closed loop system (where u^a are functions of q and t) is $\big((\mathcal{U},\mathcal{D}),\overline{\mathcal{U}}\big)$-reducible if it satisfies the conditions in Lemma 4.2.

It is also important to note that the systems response to disturbances (in closed-loop) is completely encoded in the reduction as well, precisely because we included the uncertainties in the description of the reduction. Hence, dynamic uncertainties in Eq. (3) become kinematic uncertainties in Eq. (4). This way, closed-loop design in the kinematic description are valid when implemented on the dynamic system, along with a backstepping algorithm to control the velocities of the actuators.

VII. Application to Intermittent Contact Systems

Mechanical systems that experience intermittent contact are common in engineering, and include vehicles such as the Mars Rover [14], distributed manipulation [15], [12], [2], [3], MEMS manipulation (as discussed shortly) [11], and legged locomotion [9]. To see how the result from the last section is relevant to intermittent contact systems, consider the modeling of such a system as follows. For each possible contact state, there exists a constrained affine connection ${}^G\nabla$ that encodes both the free kinetic energy and the geometric constraints (both holonomic and nonholonomic) imposed upon the system. Moreover, the external controls (corresponding to actuator control) are represented by u^aY_a (where Y_a is projected as necessary onto the distribution D). Now, if an intermittent contact system only experiences "sticking" contact or no contact, we would be done. (This is typically the case in locomotion using legged robots, for instance). If, however, the device will have surfaces sliding against each other as a result of constraints being violated, there will be additional forces coming from friction reaction forces. These forces are represented by the d^bV_b terms in Eq. (8). This situation arises when an actuation surface (such as MEMS devices or multi-point manipulation devices) experiences stick/slip phenomenon during operation. In these situations, particularly at the micro-scale, the reaction forces due to friction are not well characterized and can involve a host of friction modeling methodologies [18]. Hence, it is desirable to represent these systems in a way that does not involve the frictional reaction forces explicitly.

Note that if the contact state of a $\big((\mathcal{U},\mathcal{D}),\overline{\mathcal{U}}\big)$-reducible intermittent contact system is being driven by the frictional interactions (such as the case of MEMS manipulation discussed later), *the effects of friction are completely encoded in the σ dynamics of the reduced multiple model system.* That is, changes in contact state are expressed purely in terms of stick/slip, where for a given contact σ is acting as a binary-valued indicator function. The advantage of this is that it takes a highly nonlinear, nonsmooth phenomenon and encodes its effect as a finite state machine.

VIII. Examples

The examples discussed here illustrate how the prior results can allow one to neglect disturbance forces in mechanical systems. The first example we discuss is a MEMS scratch drive actuator (SDA) array operating on a insulating layer. After that, we discuss the control of a related macro-scale multi-point manipulator.

A. MEMS Manipulation Using Scratch Drive Actuators

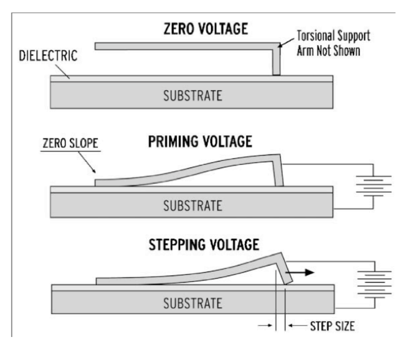

Fig. 2. Scratch Drive Actuators (SDA) (Figures taken from [11]). SDAs are chips covered with a large number of actuators along with the gold tether than is used to send voltages down to the SDAs. The figure illustrates the motion that an SDA goes through when moving.

Scratch drive actuators (SDA) are characterized by being able to produce large deflections (on the order of 500 μm), relatively large forces (on the order of 100 μN), with high precision step sizes (on the order of 30 nm). They can be arrayed on chips with as few as ten SDA actuators on a chip. Despite the fact that these devices were first developed over ten years ago [1], only recently has any formal work been done on modeling and control for these devices [11].

If one applies a voltage to an SDA, it responds by contracting. After the voltage is set back to zero, the actuator relaxes. During a sequence of such pulses, the actuator experiences intermittent nonslip contact with the underlying

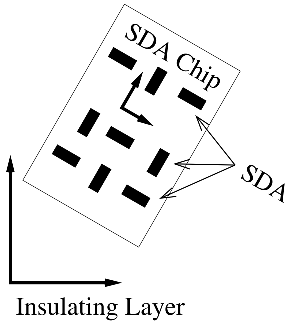

Fig. 3. Array of scratch drive actuators

insulating layer, allowing it to move in a manner similar to an inchworm. The stepping motion begins with the free end of the SDA electrostatically loaded until the threshold voltage is reached, at which point it flattens out against the insulating layer. This process is illustrated in Fig.2.

Modeling these devices depends heavily on the particulars of the brushing geometry, plate thickness, insulator properties, and the plate Young's modulus. However, an in-depth analysis of such a device was performed in [11]. The main important result of that analysis is that one can drive the actuators at a desired velocity, despite considerable uncertainty in the force characteristics. Hence, SDAs are most naturally described in terms of kinematic relationships, at least when considered individually. Solving for the forces is difficult here as well, as at the micro-scale they are typically not well defined using traditional friction models. Hence, it is desirable for any control strategy to not require this modeling and to take advantage of being able to reliably drive these actuators at a desired velocity.

Consider Fig.3. In this schematic we see a chip on an insulating layer that is actuated by nine SDAs. Each SDA is capable of moving in the direction of its long axis and is in principle constrained to not move sideways. If it does move sideways, a reaction force occurs due to the sliding. Now we ask whether such a chip can be represented as a kinematic system. Assume that the chip has mass m and rotational inertia J, so that when we write the coordinates of its body frame relative to the world as (x, y, θ), the resulting kinetic energy tensor (Riemannian metric) is $G = m\ dx \otimes dx + m\ dy \otimes dy + J\ d\theta \otimes d\theta$. For simplicity, assume that the SDA actuators are themselves of negligible mass and that they form a point contact with the insulating layer. Then, the equations of motion can be written as ${}^{G}\nabla_{c'(t)}c'(t) = u^a Y_a + d^b V_b$. In this equation the u^a correspond to each force being produced by the SDAs and the Y_a transform these forces into the body frame while respecting any constraints imposed upon the system. Such constraints arise from no-slip contact between the insulating layer and the actuators. The d^b represent reaction forces due to slipping along the insulating layer when such a constraint is violated.

Proposition 8.1: A planar array of orthogonal actuators is both $(\mathcal{U}, \mathcal{D})$-reducible and $\big((\mathcal{U}, \mathcal{D}), \overline{\mathcal{U}}\big)$-reducible.

Proof: First, it is clear that the system is $(\mathcal{U}, \mathcal{D})$-reducible. This follows from the fact that in the body frame of the chip any reaction force due to friction is a vector in $\mathbb{R}^2$, and the forces coming from the actuators span $\mathbb{R}^2$. This fact is not surprising because the system is massively overactuated, but it is unfortunately also not terribly helpful due to the fact that we cannot reliably compute the reaction forces.

We are now left with the question of whether the pictured SDA chip is $\big((\mathcal{U}, \mathcal{D}), \overline{\mathcal{U}}\big)$-reducible. Note that if we represent the chip as a rigid body with configuration in $SE(2)$, any force vector $\vec{f}$ at an actuator A_i can be represented in the inertial frame by a wrench in $se^*(2)$, namely $Ad^T_{g_{WB}g_{BA_i}}\vec{f}$. In this formula g_{WB} is the rigid body transformation from the world frame to the body frame, g_{BA_i} is the rigid body transformation from the body frame to the actuator frame, and $Ad_{g_{XY}}$ is the adjoint transformation mapping velocities in the Y frame to velocities in the X frame. (For an elementary presentation of these computations, see [17].) Hence, if we assume that forces (from constraints or from actuation) occur at the site of actuators A_i, we can compute their forces in the common inertial frame, and thus compute $u^a Y_a + d^b V_b$. Say that we choose three actuators, A_1, A_2, and A_3, each with coordinates in the body frame (a_i, b_i, ψ_i), where ψ_i are multiples of $\frac{\pi}{2}$ (since the actuators are all orthogonal). Then, if we assume that the location of the body in the world frame is (x, y, θ), the representation of each of the forces in the world frame can be written as $Ad^T_{g_{WB}g_{BA_i}}\vec{f}_i = [\cos(\theta+\psi_i), -\sin(\theta+\psi_i), y + b\cos(\theta) + a\sin(\theta)]^T$. If we take the determinant of the matrix $[Ad^T_{g_{WB}g_{BA_1}}\vec{f}_1, Ad^T_{g_{WB}g_{BA_2}}\vec{f}_2, Ad^T_{g_{WB}g_{BA_3}}\vec{f}_3]$ we find that it is nonzero as long as $\psi_1 \neq \psi_2$.

First, what does this mean if there are no constraints (i.e., no slipping orthogonal to the actuators). Then for the proper choice of actuators, $\mathbb{R}^3 = T_{(x,y,\theta)}SE(2)$ is spanned by the force vector fields $u^a Y_a$, so the system without disturbance forces is $\big(\mathcal{U}, \overline{\mathcal{U}}\big)$-reducible by Lemma 3.2 which implies that with forces it is $\big((\mathcal{U}, \mathcal{D}), \overline{\mathcal{U}}\big)$-reducible by Lemma 4.3. If there is only one constraint, then for the proper choice of input forces u^a that constraint combined with Y_a span $\mathbb{R}^3$, implying by similar logic that the dynamic system with disturbances is $\big((\mathcal{U}, \mathcal{D}), \overline{\mathcal{U}}\big)$-reducible. The argument for two constraints is identical. For three constraints, the chip is completely constrained not to move, and with more than three constraints it is kinematically overconstrained (which is physically impossible for a true rigid body and implies flexing of an elastic body). ■

Hence, an array of SDA actuators on a chip being used to make the chip move is always $\big((\mathcal{U}, \mathcal{D}), \overline{\mathcal{U}}\big)$-reducible to a kinematic system of the form in Eq. (4). Moreover, as the contact states change, the kinematic system will change. This means that the effects of friction on the dynamics of the chip are now completely encoded in the switching from one set of *kinematic* equations to another over time. This situation has well-defined control strategies, as discussed next.

B. Multi-Point Manipulation

Figure 4 represents a "top-down" simplified view of the manipulator in Fig. 1 from Section II. This is additionally the system studied in our last year's contribution to ACC [13].

Region	Equations of Motion	Control Law	Not Slipping
I	$\dot{q} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} u_2$	$u_1 = \frac{-u_2\,(\theta+x-y)+k\,(\theta^2+x^2+y^2)}{x+y}$, $u_2 = -k\theta$	1S 1R 4R
II	$\dot{q} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} u_2$	$u_1 = \frac{-u_2\,(\theta+x-y)+k\,(\theta^2+x^2+y^2)}{x+y}$, $u_2 = -k\theta$	1S 1R 2R
III	$\dot{q} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} u_2$	$u_1 = k\theta$, $u_2 = \frac{u_1\,(\theta+x+y)-k\,(\theta^2+x^2+y^2)}{x-y}$	1R 2R 2S
IV	$\dot{q} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} u_2$	$u_1 = k\theta$, $u_2 = \frac{u_1\,(\theta+x+y)-k\,(\theta^2+x^2+y^2)}{x-y}$	2R 2S 3R
V	$\dot{q} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} u_2$	$u_1 = \frac{u_2\,(\theta-x+y)-k\,(\theta^2+x^2+y^2)}{x+y}$, $u_2 = k\theta$	2R 3R 3S
VI	$\dot{q} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} u_2$	$u_1 = \frac{u_2\,(\theta-x+y)-k\,(\theta^2+x^2+y^2)}{x+y}$, $u_2 = k\theta$	3R 3S 4R
VII	$\dot{q} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} u_2$	$u_1 = -k\theta$, $u_2 = \frac{u_1\,(-\theta+x+y)+k\,(\theta^2+x^2+y^2)}{x-y}$	3R 4R 4S
VIII	$\dot{q} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} u_2$	$u_1 = -k\theta$, $u_2 = \frac{u_1\,(\theta-x-y)-k\,(\theta^2+x^2+y^2)}{x-y}$	4R 4S 1R

TABLE I

LIST OF REGIONS, EQUATIONS OF MOTION, CONTROL LAWS, AND ACTIVE CONSTRAINTS. THE REGIONS ARE DENOTED AS **I-VIII** AS IN FIG. 4. THE WHEELS ARE LABELED ACCORDING TO THEIR QUADRANT (I.E., WHEEL 1 IS IN QUADRANT 1,). ACTIVE (I.E., NONSLIPPING) CONSTRAINTS IN THE ROLLING DIRECTION ARE LABELED WITH R, WHILE ACTIVE CONSTRAINTS IN THE SIDEWAYS DIRECTIONS ARE LABELED WITH S (I.E., 1R INDICATED THAT WHEEL 1 DOES NOT SLIP IN ITS ROLL DIRECTION).

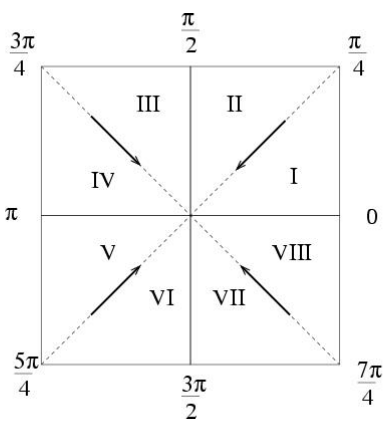

Fig. 4. Schematic of four actuator system in Figure 1.

This figure has four actuators (corresponding to the inputs $u_1, \ldots, u_4$ and represented in the figure by arrows) located at $(1,1), (-1,1), (-1,-1), (1,-1)$ respectively, all pointed towards the origin, just as in the physical system. An analysis of this system, identical to the one just shown of the MEMS system, shows that this system is kinematic. If one uses Coulomb friction for nominal analysis, as we did in [15], it turns out that the plane can be divided into 8 distinct regions, labeled $\mathbf{I}-\mathbf{VIII}$, where one contact state holds. These are separated by 8 boundaries, labeled $0-2\pi$ in increments of $\frac{\pi}{4}$. If we assume that a continuous control can be applied at each one of these actuator locations, one can design a controller as follows. In each one of the regions $\mathbf{I}-\mathbf{VIII}$ a control law is calculated from the Lyapunov function $k(x^2+y^2+\theta^2)$ by solving $\dot{V} = -V$ for u_i, where k is some constant to be chosen during implementation. Therefore, there are eight *static* control laws, each defined in a separate octant, and none of which depend on the friction model.

Table I shows the equations of motion and control law for each octant as well as indicating which wheels are predicted to be slipping. The control laws satisfy $u_3 = -u_1$ and $u_4 = -u_2$, so only u_1 and u_2 are listed in the control specification. Hence in region **I** the satisfied constraints are wheel 1 not slipping in either direction (1R 1S) and wheel 4 not slipping in the rolling direction (4R). These three constraints then uniquely determine the equations of motion. Note that these control laws are not only nonlinear, they are not even smooth. In fact, they have discontinuities that coincide with the boundaries and, in particular, discontinuities at the origin (the point towards which we are stabilizing).

Figure 5 shows a simulation of the four point manipulation system depicted in Fig. 1. This simulations was done in *Mathematica*, using Euler integration in order to avoid numerical singularities when crossing contact state boundaries. The object is indicated by a rectangle, but the reader should note that although the rectangle is illustrated as being small, the actual body it represents is in contact with all four actuators at all times, which are denoted in the figure by Nodes 1-4. The initial condition is $\{x_0, y_0, \theta_0\} = \{.5, 2, \frac{\pi}{2}\}$, and progress in time is denoted by the lightening of the object. The goal is to stabilize the system to the origin of $SE(2)$, $\{x_f, y_f, \theta_f\} = \{0, 0, 0\}$. The simulation shows

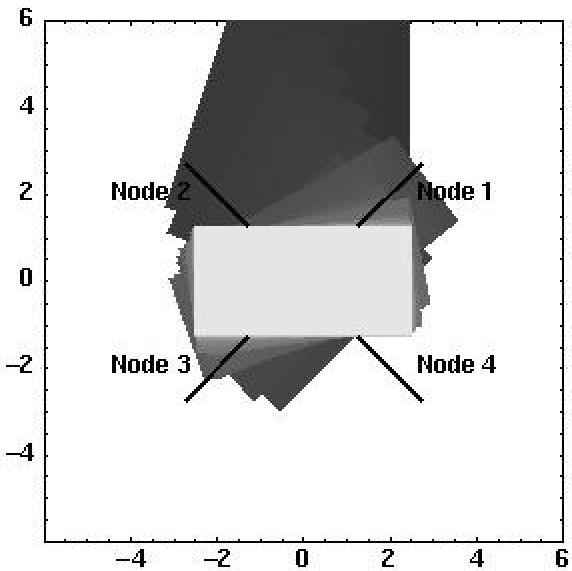

Fig. 5. Simulation of multi-point manipulation using the control laws from Table I.

that the system can be stabilized even with the changing contacts, and without modeling friction explicitly. Moreover, this trajectory is qualitatively very similar to the trajectories found experimentally in [15].

IX. Conclusion

In this paper we examined the problem of modeling multiple contact systems in such a way that the control design can avoid sensitivity to particular models of friction. To this end, sufficient conditions for a mechanical system with external disturbances to be reduced to a kinematic system were obtained. This work illustrates how the effects of friction, which at the micro-scale in particular are very complex and nonlinear, can be encoded into a hybrid structure that does not require explicit modeling of friction. A simple model of a MEMS SDA array was then analyzed and a macro-scale multiple contact system was used to illustrate the control design procedure one may employ. In future work we plan to build on this foundation to use SDA actuators in micro-scale assembly.

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