Geometric Derived Information Spaces in Manipulation with Mechanical Contact

Todd D. Murphey

***Abstract*— This paper describes methods applicable to the modeling and control of mechanical contact, particularly those that experience uncertain stick/slip phenomena. Geometric kinematic reductions are used to show how to reduce a system's description from a second-order dynamic model with frictional disturbances coming from a function space to a first-order model with frictional disturbances coming from a space of finite automata over a finite set. As a result, modeling for purposes of control in the resulting derived information space is made more straight-forward by getting rid of some dependencies on low-level mechanics (in particular, the details of friction modeling). Moreover, the online estimation of the uncertain variables in the derived information space has reduced sensing requirements. Results are illustrated using an actuator array model.**

I. INTRODUCTION

It is traditional in robotics to view problems of manipulation, motion planning, and control in one of two extreme lights. First, if a system is kinematic, the system description is simplified from a second-order system with forces and inertias to a first-order system that consists of velocities and constraints. Then motion plans and control laws (if necessary) are designed for this kinematic system. It is important to note that in order to *implement* this plan based on kinematics, a backstepping algorithm is employed, either explicitly in an "inner-loop-outer-loop" control architecture, or implicitly by purchasing motor controllers (or other appropriate devices) that provide the inner loop control. In the end, the advantages of using kinematic structures include both lessened computational burden (due to the computation in a lower-dimensional space) and increased robustness to some classes of uncertainty (due to robustness properties of the backstepping, inner-loop controller).

If, however, there is some reason that a kinematic analysis is inappropriate, then one often reverts to a more complex set of modeling choices. In particular, in multi-point contact many phenomena are introduced, including soft-contact models [2], elaborate models of frictional interfaces [15], and the inclusion of dynamic effects such as inertial terms and generalized forces. Nevertheless, it is not clear that the introduction of these additional modeling techniques helps for the purpose of control, motion planning, etcetera. In particular, the task description typically does not include these effects, so one should only incorporate them in the representation in use for planning and control if they are actually necessary for task completion (which they typically are not).

From a design perspective (as opposed to a simulation perspective), it is thus desirable to, if necessary, introduce elements to a model that provide the full complexity of possible behavior of the system without introducing too much new information (thereby decreasing the applicability of the model). This is related to *information spaces* [8], which originated in the computer science community and were then formalized in a control abstraction context in [16] and related works. In this paper, notions of an information space $\mathcal{I}$ and an associated (in some sense smaller) derived information space $\mathcal{I}_{der}$, along with their desired properties, play a central role. In particular, we are interested in understanding when planning and control problems can be successfully posed in a smaller, more tractable space.

This paper shows how notions of kinematic reducibility can allow one to recast a dynamic system that has frictional effects belonging to a function space into a first-order system that has frictional effects that form a finite automaton over a finite set. This provides a representation of friction that is simultaneously more simple and less naive (in the sense that one no longer needs to know *which* friction law is governing the dynamic equations of motion for purposes of implementation).

The basic thesis of this paper is that kinematic systems are useful not so much because of their first-order representation of a system, but rather because they arise naturally from a canonical choice of derived information space $\mathcal{I}_{der}$. This derived information space allows one to get rid of most (but not all) of the information about the frictional interactions. The key contribution of this paper is the presentation of a methodology for creating a derived information space $\mathcal{I}_{der}$ based on geometric principles. In particular, the goal is to create a reduced space that preserves both trajectories and stability, so that any plan or controller in $\mathcal{I}_{der}$ will have a stable implementation in $\mathcal{I}$. Surprisingly, orthogonal actuator arrays (the example we dicuss) are *always* kinematic. This is true even if all contact points are slipping against the surface of a manipulated object, so long as the frictional interaction is strictly dissipative.

This paper is organized as follows. Section II describes micro-mechanical scratch drive actuators that motivate the present work. Section III discusses information spaces and why one may wish to solve problems in a derived information space. Section IV discusses modeling of multi-point contact systems using the constrained affine connection and Section V discusses some straight-forward results in kinematic

The author gratefully acknowledges the support for this work provided by the NSF under CAREER award CMS-0546430.

Todd D. Murphey is with Faculty of Electrical and Computer Engineering, University of Colorado at Boulder, Boulder, CO, USA murphey@colorado.edu

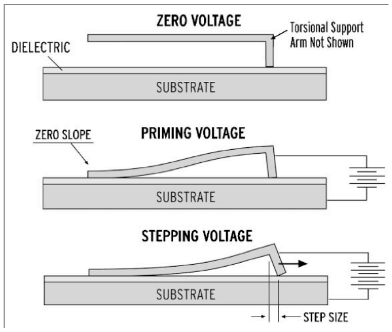

Fig. 1. Scratch Drive Actuators (SDA) (Figure taken from [10]). SDAs are chips covered with a large number of actuators along with the gold tether than is used to send voltages down to the SDAs. The figure illustrates the motion that an SDA goes through when moving. Despite being able to drive SDA actuators quite reliable, the individual forces are difficult to model accurately.

reduction for systems with external forces. Because control and estimation are occurring directly in the derived information space, Section VI discusses the method employed for estimating discrete variables in $\mathcal{I}_{der}$. Section VII discusses the application of the tools presented to actuator arrays.

II. Motivation: Mechanical Contact Systems

A system consisting of many points of contact typically exhibits stick/slip phenomenon due to the point contacts moving in kinematically incompatible manners. This manner of motion is called *overconstrained motion* because not all of the constraints can be satisfied.

Consider the example in Fig. 1. Scratch drive actuators (SDA) are characterized by being able to produce large deflections (on the order of 500 μm), relatively large forces (on the order of 100 μN), with high precision step sizes (on the order of 30 nm). They can be arrayed on chips with as few as ten SDA actuators on a chip. Despite the fact that these devices were first developed over ten years ago [1], only recently has any formal work been done on modeling and control for these devices [10].

If one applies a voltage to an SDA, it responds by contracting. After the voltage is set back to zero, the actuator relaxes. During a sequence of such pulses, the actuator experiences intermittent nonslip contact with the underlying insulating layer, allowing it to move in a manner similar to an inchworm. The stepping motion begins with the free end of the SDA electrostatically loaded until the threshold voltage is reached, at which point it flattens out against the insulating layer. This process is illustrated in Fig. 1.

Modeling these devices depends heavily on the particulars of the brushing geometry, plate thickness, insulator properties, and the plate Young's modulus. An in-depth analysis of such a device was performed in [10]. The main important result of that analysis is that one can drive the actuators at a desired velocity, despite considerable uncertainty in the force characteristics. Hence, SDAs are most naturally described in terms of kinematic relationships, at least when considered individually. Solving for the forces is difficult here as well, as at the micro-scale they are typically not well defined using traditional friction models. Hence, it is desirable for any control strategy to not require this modeling and to take advantage of being able to reliably drive these actuators at a desired velocity.

III. Information Spaces in Mechanical Contact

The notion of an information space $\mathcal{I}$ and an associated derived information $\mathcal{I}_{der}$ will be helpful in a discussion of mechanical contact. In particular, a *choice* of $\mathcal{I}_{der}$ has profound implications for both analysis, problem decomposition, and implementation of control in systems with mechanical contact.

Sensors (and the underlying dynamics of the physical system) induce what is termed an *information space* [8]. Typically, an information space, $\mathcal{I}$, consists of the time history of measurements and the control history of actions. That is, the information space is $\mathcal{I} = (y(t), u(t))$ where $y(t)$ is the history of discrete measurements (or, possibly, the continuous space of measurements), and $u(t)$ is similarly the history of inputs. Additionally, one can choose a smaller information space that retains only the critical information in $\mathcal{I}$, and is therefore simpler, but more of an abstraction; this is called the *derived information space* $\mathcal{I}_{der}$. The derived information space $\mathcal{I}_{der}$ depends on task specification and the associated sensing needs: different tasks will have a differing subset of critical information coming from $\mathcal{I}$. Hence, $\mathcal{I}_{der} = (\tilde{y}(t), \tilde{u}(t))$ where $\tilde{y}(t)$ and $\tilde{u}(t)$ are the set of measurements and inputs after they have been manipulated in some way to reduce their complexity.

Given a "low-level" information space $\mathcal{I}$ and a "high-level" derived information space $\mathcal{I}_{der}$, one can compute a plan in the latter and implement it in the former. Hence, such a plan is a mapping $\pi : \mathcal{I}_{der} \to \mathcal{I}$ (where $\pi \in \Pi$, the space of all such mappings). Again, depending on the needs of the problem $\mathcal{I}_{der}$ may have different structure. For a planning problem, $\mathcal{I}_{der}$ may be a path in the configuration, or simply a set of waypoints. For control, one may wish to design a controller in $\mathcal{I}_{der}$ and have it maintain its stability when implemented in $\mathcal{I}$. In each case, the mapping $\pi : \mathcal{I}_{der} \to \mathcal{I}$ needs to implement commands without violating stability and other dynamic characteristics of the low-level space.

The information space underlying mechanical contact will be of the form $\mathcal{I} = ((x(t), \tau(t)), u(t))$ with $x(t) \in TQ$, Q the configuration manifold, TQ the tangent bundle, $\tau(t) \in V$ where V is a set of values the friction reaction forces can take and $\tau \in \mathcal{L}^r$ the space of r integrable functions for r contacts (this will be restricted slightly, since f will correspond to a friction law), and $u(t) \in U$ where U is the space of control forces that can be applied to the system. Hence, $\mathcal{I}$ is simply an estimate of the state along with an estimate of the friction law and a corresponding control action. One would like to be able to compute both planning and feedback laws in a simpler space–one that does not require characterizing $\tau \in \mathcal{L}$.

It will be shown that a choice of $\mathcal{I}_{der} = ((q(t), \sigma(t)), \tilde{u}(t))$ where $q \in Q$, $\sigma(t) \in \Sigma$ (Σ a finite set), and $\tilde{u}(t) \in \tilde{U}$ (velocity inputs for a kinematic system) is a derived information space that preserves trajectories (in the sense that trajectories in $\mathcal{I}_{der}$ always represent trajectories in $\mathcal{I}$ and every trajectory in $\mathcal{I}$ has a representation in $\mathcal{I}_{der}$) if it is $\big((\mathcal{U},\mathcal{D}),\overline{\mathcal{U}}\big)$-reducible (defined shortly) and can be implemented in a stable manner (in the sense that a stabilizing controller designed in $\mathcal{I}_{der}$ can be mapped to a stable controller in $\mathcal{I}$). Hence, in this setting π is a backstepping controller that implements kinematic inputs in the dynamic space.

IV. Modeling and Analysis of Multiple Point Contact

The systems considered here are finite-dimensional simple mechanical systems (as described for smooth systems in [3]). That is, their equations of motion may be found using a Lagrangian of the form kinetic energy minus potential energy ($L = K.E. - V$) along with a set of constraints on the system of the form $\omega(q)\dot{q} = 0$, where $\omega(q)$ is a matrix representing the configuration q dependent constraints. Moreover, there may be external forces acting on the system. If one ignores potential energy (as is appropriate for many planar systems including the one described in Section II), such a system's dynamics may be represented as:

$$\nabla_{\dot{q}}\dot{q} = u^\alpha Y_\alpha, \qquad (1)$$

where the notation $u^\alpha Y_\alpha$ implies summation over the α. In this expression, ∇ is the constrained affine connection encoding the free kinetic energy and any constraints on the system. Moreover, u represents external forces (not necessarily inputs) and Y represents the associated vector fields on the configuration manifold Q (i.e., $Y \in T_qQ$, the tangent space at $q \in Q$). If one wishes to include potential energy, it will show up as a vector field on the right-hand side of the equation. A short description of this formulation of mechanics may be found in the Appendix.

The systems of interest have two types of external forces–those that correspond to inputs and those that correspond to external disturbances. In the case of multiple point contact, the external disturbance forces generally correspond to reaction forces due to friction when a contact slips. Therefore, it will be useful to write the dynamic equations as: $\nabla_{\dot{q}}\dot{q} = u^\alpha Y_\alpha + d^\beta V_\beta$ so that a distinction between external forces that can be controlled and those that cannot can be made.

A. Standing Assumptions on Friction

Consider some of the standard friction models, seen in Fig. 2. These of course include Coulomb friction ($F = F_C sign(v)$ for $F_C > 0$), but additionally include viscous friction, stiction, and nonlinear versions, such as a better representation of viscous friction. These are respectively represented as

$$F = \begin{cases} F_V v + c & v > 0 \\ (-c, c) & v = 0 \\ F_V v - c & v < 0 \end{cases} \qquad F = \begin{cases} F_V v + c & v > 0 \\ (-c-\delta, c+\delta) & v = 0 \\ F_V v - c & v < 0 \end{cases}$$

$$F = \begin{cases} F_V |v|^{\delta_c} sign(v) + c & v > 0 \\ (-c, c) & v = 0 \\ F_V |v|^{\delta_c} sign(v) - c & v < 0 \end{cases}$$

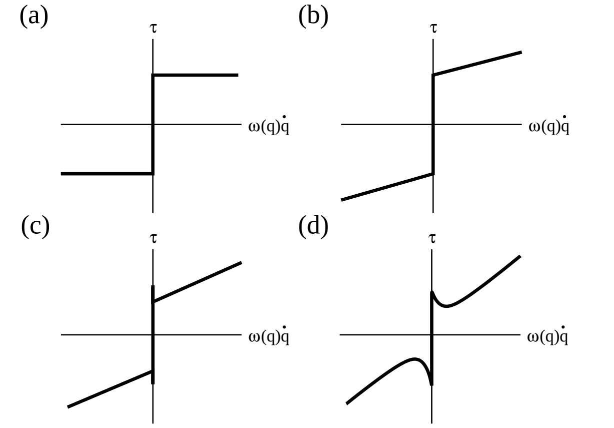

Fig. 2. Types of friction model, including (a) Coulomb friction, (b) Coulomb plus viscous friction, (c) Coulomb/viscous stiction, and (d) Nonlinear smoothing of stiction.

for $F_V, c, \delta > 0$. These are seen in Fig. 2. Moreover, there are many more types of friction model to choose from, including dynamic models of friction like Dahl and LuGre models [15] or even more heuristic models such as Pacejka's "Magic Tire Formula"–each with their own specialized area of applicability. What one would like is an assumption on friction that does not depend on any of these particular characteristics. Although they are qualitatively similar to each other, we would like to conservatively bound the class of friction models and choose a derived information space that is invariant with respect to the particular friction model. That is, one would like to know that for any admissible friction model and any parameters for those friction models, the dynamic mapping $\pi : \mathcal{I}_{der} \to \mathcal{I}$ is a stable implementation in $\mathcal{I}$.

With this goal in mind, replace the family of curves seen in Fig. 2 by the conservative estimation of those curves seen in Fig. 3. In this figure, the friction law need only be dissipative. That is, $v > 0 \Rightarrow \tau > 0$ and $v < 0 \Rightarrow \tau < 0$. If $v = 0$, then $\tau \in \mathbb{R}$–that is, stiction (constraint) forces are allowed, and frictional constraints are allowed. (This is the first time any use for the constrained affine connection becomes apparent.) The important thing to note is that $v \neq 0 \Rightarrow \tau \neq 0$–this will be important later. In any case, the friction curve can be any absolutely continuous curve that has all its values in the grayed regions in Fig. 3.

(Ultimately τ will restricted slightly more for purposes of stability analysis.) Hence, if $\omega(q)\dot{q}$ is the slipping velocity at some point, we restrict τ in the following manner.

$$\tau(\omega(q)\dot{q}) = \begin{cases} \tau(\omega(q)\dot{q}) > 0 & \text{if } \omega(q)\dot{q} > 0 \\ \tau(\omega(q)\dot{q}) < 0 & \text{if } \omega(q)\dot{q} < 0 \\ \tau(\omega(q)\dot{q}) \in \mathbb{R} & \text{if } \omega(q)\dot{q} = 0 \end{cases} \qquad (2)$$

With this picture in mind, one can now *choose* an equivalence class on $\tau \in \mathcal{L}$ that will be familiar. In particular, let us consider the cases $\omega(q)\dot{q} = 0$ (when the system is constrained) and $\omega(q)\dot{q} \neq 0$ (when the system is sliding) separately. *That is, we arbitrarily choose to distinguish between* slipping *friction forces and* constraint *friction forces.*

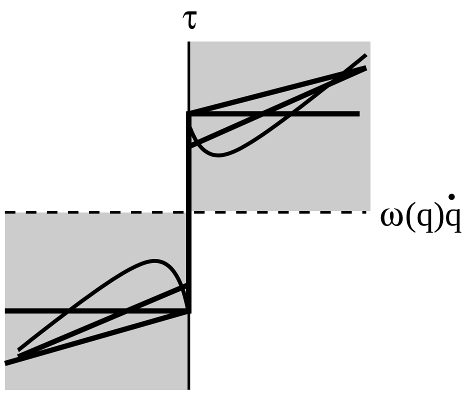

Fig. 3. Friction is only assumed to be dissipative, so that any curve in the grayed areas is a valid friction model. Clearly, this includes all the friction models in Fig. 2 (shown again here) and more[15].

This canonical distinction is traditionally referred to as the *contact state* of a system.

In particular, when $\omega(q)\dot{q} = 0$, the dynamics may still be written as $\nabla_{\dot{q}}\dot{q} = u^{\alpha}Y_{\alpha}$, where ∇ is now the constrained affine connection and Y_{α} are appropriately projected onto the distribution (see Appendix). Moreover, because the contact state changes over time (as the contacts transition between stick and slip), the constraints change over time. This implies that ∇ is not a single constrained affine connection, but rather comes from a set of constrained affine connections ∇^{σ}, each of which represents a different set of stick/slip states of the mechanism. The same holds true for Y^{σ} and V^{σ}. Hence, if one indexes the set of possible stick/slip states by σ, one gets second-order equations of motion of the following form:

$$\nabla^{\sigma}_{\dot{q}}\dot{q} = u^{\alpha}Y^{\sigma}_{\alpha} + d^{\beta}V^{\sigma}_{\beta} \tag{3}$$

where u are input forces and d are external forces. Reducing Eq. (3) to a first-order description without friction and retaining σ as the representation of frictional effects is the focus of Section V and will create the derived information space $\mathcal{I}_{der}$.

V. Kinematic Reduction With External Forces

We now take a slight departure from discussing information spaces directly and focus on kinematic reductions [4], [6], [5], [9], [14], [3]. Smooth kinematic reductions take systems of the form of Eq. (1) and convert them into systems of the form

$$\dot{q} = \overline{u}^{a}X_{a}. \tag{4}$$

The affine connection formalism in Section IV is used to describe mechanical systems because it is in the context of this formalism that a useful technical connection between 2[nd]-order mechanical systems and 1[st]-order kinematic systems has been made (found for smooth systems in [9] and for nonsmooth systems in [14]). In particular, it would be useful to be able to write Eq. (3) in the form:

$$\dot{q} = \overline{u}^{a}X^{\sigma}_{a}, \tag{5}$$

where $\overline{u}$ are *velocity* inputs instead of force inputs and σ is allowed to switch the vector fields X discretely just as it does in Eq. (3). What is shown here is that the algebraic test of kinematic reducibility (in the presence of switching in σ and external forces $d^{b}V_{b}$) is that the *symmetric product* between two vector fields Y^{σ}_{i} and Y^{σ}_{j} (defined by $\left\langle Y^{\sigma}_{i} : Y^{\sigma}_{j}\right\rangle = \nabla^{\sigma}_{Y^{\sigma}_{i}}Y^{\sigma}_{j} + \nabla^{\sigma}_{Y^{\sigma}_{j}}Y^{\sigma}_{i}$ for given i, j, σ) lie within the distribution of the vector fields and that any reaction forces lie within the span of the input vector fields. That is, a system is kinematically reducible if and only if the following conditions hold.

$$\begin{aligned} \left\langle Y^{\sigma}_{i} : Y^{\sigma}_{j}\right\rangle &\in \text{span}_{\mathbb{R}}\{Y_i | i = 1, \dots, m\} \quad \forall\, i, j, \sigma \quad (6)\\ V^{\sigma}_{\beta} &\in \text{span}_{\mathbb{R}}\{Y_i | i = 1, \dots, m\} \quad \forall\, \beta, \sigma. \quad (7) \end{aligned}$$

This result is the focus of the rest of this section.

A. Reduction for single model systems

Initially reduction for single model systems of the following form is considered.

$$\nabla_{c'(t)}c'(t) \in u^{a}(t)Y_{a}(c(t)) + d^{b}(t)V_{b}(c(t)) \tag{8}$$

In this equation ∇ is the (possibly constrained) affine connection associated with the Riemannian metric G, d^{b} is a set of forces corresponding to external disturbances that meet the assumptions in Section IV-A in Eq. (2), V_{b} is the set of corresponding vector fields, u^{a} is a set of forces corresponding to control inputs, and Y_{a} are the associated vector fields. Since the motivation here is not wanting to be forced to rely on the correctness of one particular disturbance force model (such as friction force modeling where there are many possible choices of model), the term $d^{b}V_{b}$ is assumed to be set-valued for each index b, as in Fig. 3. If $d^{b}V_{b}$ as a set is not convex, then it is replaced by its convex hull $co\{d^{b}V_{b}\}$ so as to guarantee solutions exist in the Filippov sense [7].

Now, given a system with set-valued disturbances such as in Eq. (8), under what circumstances it can be reduced to a system of the form in Eq. (1)? That is, when can one find an equivalent system that does not include external disturbance forces. To make such an equivalence more rigorous, we introduce some definitions, following the Appendix for guidance.

Definition 5.1: Let Σ_{s} be a smooth control system $\dot{q} = f(q, u, d)$ on a smooth manifold M. A $(\mathcal{U}, \mathcal{D}, \mathcal{T})$-*solution* to Σ_{s} is a triple (c, u, d), where $u : [0, T] \to U_{\text{input}} \subseteq \mathbb{R}^{m}$, $d : [0, T] \to U_{\text{disturbance}} \subseteq \mathbb{R}^{l}$, and $c : [0, T] \to M$ satisfy $c'(t) = f(c(t), u(t), d(t))$.

We now define the following notion of reduction, which simply requires that solutions in the reduced space always correspond directly to solutions in the original space.

Definition 5.2: Let ∇ be an affine connection on Q, and let $\mathcal{U}$ be a family of control functions and $\mathcal{D}$ be a family of disturbance functions. The system in Eq. (8) is $(\mathcal{U}, \mathcal{D})$-*reducible* to the system in Eq. (1) if for each $(\mathcal{U}, \mathcal{D}, \mathcal{T})$-solution (η_1, u_1, d) of the Eq. (8) there exists a $(\mathcal{U}, \mathcal{T})$-solution (η_2, u_2) of Eq. (1) with $\eta_1(t) = \eta_2(t)$;

Lastly, one would like to be rigorous about what it means for a mechanical system with set-valued disturbances to be reducible to a kinematic system, which leads to the following definition.

Definition 5.3: Let ∇ be an affine connection on Q, and let $\mathcal{U}$ and $\overline{\mathcal{U}}$ be two families of control functions. The system in Eq. (8) is $\left((\mathcal{U},\mathcal{D}),\overline{\mathcal{U}}\right)$*-reducible* to the system in Eq. (4) if the following two conditions hold:

i) for each $(\mathcal{U},\mathcal{D},\mathcal{T})$-solution (η,u,d) of the dynamic Eq. (1) with initial conditions $\eta(0)$ in the distribution D_{kin}, there exists a $\left(\overline{\mathcal{U}},\mathcal{T}\right)$-solution $(\gamma,\overline{u})$ of the kinematic Eq. (4) with the property that $\gamma=\tau_Q\circ\eta$;

ii) for each $\left(\overline{\mathcal{U}},\mathcal{T}\right)$-solution $(\gamma,\overline{u})$ of the kinematic Eq. (4), there exists a $(\mathcal{U},\mathcal{D},\mathcal{T})$-solution (η,u,d) of the dynamic Eq. (1) with the property that $\eta(t)=\gamma'(t)$ for almost every $t\in[0,T]$.

With these definitions, we can state sufficient conditions for $(\mathcal{U},\mathcal{D})$-reducibility and for $\left((\mathcal{U},\mathcal{D}),\overline{\mathcal{U}}\right)$-reducibility. Intuitively, this corresponds to being able to guarantee that any solutions that include disturbances can be mapped directly to a solution that has no disturbances.

Lemma 5.1: Assume one has a mechanical system of the form in Eq. (8) with unbounded inputs and dissipative friction forces τ as in Eq.(2). Then the system in Eq. (8) is $(\mathcal{U},\mathcal{D})$-reducible to the mechanical system in Eq. (1) iff this system satisfies $co\{d^bV_b\}\in \text{span}_{\mathbb{R}}\{Y_a\}$ for all b.
(The proof of this and other results in this paper are ommitted for brevity.)

This means that all trajectories can be planned as if there are no forces due to the terms d^bV_b. However, it is important to note that the requirement that $u^a\neq 0$ is satisfied precisely because we do not allow $\tau\neq 0$ for $v\neq 0$.

We are now interested in finding out when a multiple model of the form in Eq.(3) is reducible to a system of the form in Eq. (5).

Theorem 5.2: Equation (3) is $\left((\mathcal{U},\mathcal{D}),\overline{\mathcal{U}}\right)$-reducible iff Equation (3) is $\left((\mathcal{U},\mathcal{D}),\overline{\mathcal{U}}\right)$-reducible for every constant σ (i.e., Eqs. (6) and (7) hold for any choice of σ).
To sum up, if a system of the form in Eq. (3) satisfies the algebraic conditions in Eqs. (6) and (7), the system can be represented as a kinematic system and planning and control can take place in $\mathcal{I}_{der}$ without any loss of trajectory information.

B. Kinematic Reductions in Closed Loop

Everything discusses so far has implicitly relied on the control being "open-loop." However, if one is using a discrete time controller (with one's favorite continuation algorithm, such as zero-order holds) the control is open loop in between controller updates. It was already shown in [14] that $\left(\mathcal{U},\overline{\mathcal{U}}\right)$-reductions are not affected by occasional discontinuities. By the exact same logic, the systems considered here are reducible in discrete time closed-loop if they satisfy the requirements to be reducible under the tests of Lemma 6.1

Lemma 5.3: A discrete-time closed loop system (where u^a are functions of q and t) is $\left((\mathcal{U},\mathcal{D}),\overline{\mathcal{U}}\right)$-reducible if it satisfies the conditions in Lemma 6.1.

It is also important to note that the systems response to disturbances (in closed-loop) is completely encoded in the reduction as well, precisely because we included the uncertainties in the description of the reduction. Hence, dynamic uncertainties in Eq. (1) become kinematic uncertainties in Eq. (4). This way, closed-loop design in the kinematic description are valid when implemented on the dynamic system, along with a backstepping algorithm to control the velocities of the actuators.

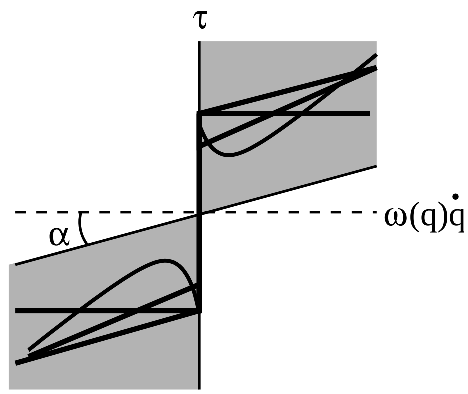

Fig. 4. An additional requirement is that the friction curve lie within a sector nonlinearity that allows the use of a proportional controller in implementation.

We need the plan $\pi:\mathcal{I}_{der}\to\mathcal{I}$ to be a stable implementation. We change the assumption on τ in Eq. (2) slightly by requiring that the reaction force curve must lie in the grayed area in Fig. 4, where $\alpha>0$. Then a choice of backstepping controller

$$u_i=-K(v_i-\overline{u}_i)+d_i \tag{9}$$

provides a stable response because the grayed region is a sector nonlinearity [17]. (This has already been used in the analysis of multi-point contact in [11].) Also, note that the use of a sector nonlinearity also allows us to take into account dynamic shifts in normal force without any extra analysis.

VI. Estimation in $\mathcal{I}_{der}$

If one wishes to design a plan or control of some sort in $\mathcal{I}_{der}$, then online estimation of σ may be necessary. Suppose for any σ we have a stable estimator of $q\in Q$ such that there is a quadratic Lyapunov function V_σ in the error of the state. Then a reasonable estimate of σ (which we will denote $\hat{\sigma}$) could evolve according to

$$\mathbb{E}(y)=\arg\min_{\sigma}\|\tilde{y}_\sigma-\tilde{y}\|$$

where $\tilde{y}_\sigma$ is the expected output for each σ and $\tilde{y}$ is the measured output. However, this estimate may be poor because it may not be stable as σ changes in time. Hence, an adjustment is necessary to estimate both q and σ in $\mathcal{I}_{der}$.

In order to create a stable estimate of σ, we first define some useful notation. First, define
$s(t)=\lim_{\tilde{t}\to t^-}V_\sigma(\tilde{t})-\lim_{\tilde{t}\to t^+}V_\sigma(\tilde{t})$. This is the discrete change in the value of the Lyapunov function for the estimator that occurs when there is a switch in σ. Next define

$$\begin{array}{rcll}\dot{E}(t)&=&-k_ed(t) & \text{if } s(t)=0\\ E(t)&=&\lim_{\tilde{t}\to t^-}E(\tilde{t})-s(t) & \text{otherwise}\end{array} \tag{10}$$

where d is a bounded conservative estimate of the stability margin for all the estimators and where k_e is a chosen constant, $0 < k_e < 1$. Note that E is initialized to a nonnegative value and then evolves according to Equation 10 as long as s is zero (that is, on intervals with no switches). Whenever $s \neq 0$ (there is a switch), E is re-initialized. Then we use the following equation to estimate σ.

$$\hat{\sigma}(t) = \begin{cases} \mathbb{E}(\tilde{y}(t)) & \text{if } E > 0 \text{ for all } i \\ \lim_{\tilde{t} \to t^-} \mathbb{E}(\tilde{y}(t)) & \text{otherwise} \end{cases} \tag{11}$$

Theorem 6.1: An estimate of $\hat{\sigma}$ using Eq. (11) is stable. That is, $|\mathbf{V}_{\sigma'(t)} - \alpha| \to 0$ for some $\alpha \in \mathbb{R}$ and, in particular, $\dot{\mathbf{V}}_{\sigma'(t)} \to 0$. Moreover, $|\hat{\sigma(t)} - \sigma(t)| = 0$ after a finite amount of time if σ is constant.

VII. Example

Note that if the contact state of a $\left((\mathcal{U}, \mathcal{D}), \overline{\mathcal{U}}\right)$-reducible intermittent contact system is being driven by the frictional interactions (such as the case of MEMS manipulation), *the effects of friction are completely encoded in the σ dynamics in $\mathcal{I}_{der}$*. The advantage of this is that it takes a highly nonlinear, nonsmooth phenomenon and encodes its effect as a finite state machine. The example discussed here illustrates how the prior results can allow one to neglect disturbance forces in mechanical systems.

Consider Fig. 5. In this schematic we see a chip on an insulating layer that is actuated by nine SDAs (discussed in Section II). Each SDA is capable of moving in the direction of its long axis and is in principle constrained to not move sideways. If it does move sideways, a reaction force occurs due to the sliding. Such a chip can be viewed as a micro-scale vehicle capable of "driving" on the insulating layer [10]. Now we ask whether such a chip can be represented as a kinematic system.

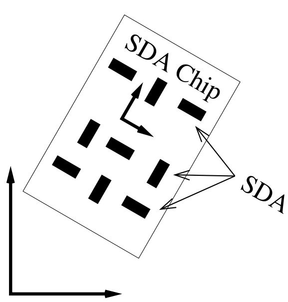

Fig. 5. Array of scratch drive actuators

Assume that the chip has mass m and rotational inertia J, so that when we write the coordinates of its body frame relative to the world as (x, y, θ), $G = m\ dx \otimes dx + m\ dy \otimes dy + J\ d\theta \otimes d\theta$. The information space $\mathcal{I} = (y, u)$ has $y \in TSE(2) \times \mathcal{L}^r$, where r is the number of actuators. The derived information space will be $\mathcal{I}_{der} = (\tilde{y}, \tilde{u})$ with $\tilde{y} \in SE(2) \times \Sigma$ and Σ is a finite set that describes the total number of kinematic states for the system. For simplicity, assume that the SDA actuators are themselves of negligible mass and that they form a point contact with the insulating layer. Then, the equations of motion can be written as $\nabla_{c'(t)} c'(t) = u^a Y_a + d^b V_b$. In this equation the u^a correspond to each force being produced by the SDAs and the Y_a transform these forces into the body frame while respecting any constraints imposed upon the system. Such constraints arise from no-slip contact between the insulating layer and the actuators. The d^b represent reaction forces due to slipping along the insulating layer when such a constraint is violated. We now analyze whether a planar array of alternately orthogonal actuators (such as those seen in Fig. 5) is kinematic.

Proposition 7.1: An object manipulated by a planar array of alternatively orthogonal actuators has dynamics that are both $(\mathcal{U}, \mathcal{D})$-reducible and $\left((\mathcal{U}, \mathcal{D}), \overline{\mathcal{U}}\right)$-reducible.

Hence, an array of actuators manipulating an object is always $\left((\mathcal{U}, \mathcal{D}), \overline{\mathcal{U}}\right)$-reducible to a kinematic system of the form in Eq. (4). Moreover, as the contact states change, the kinematic system will change. This means that the effects of friction on the dynamics of the chip are now completely encoded in the switching from one set of *kinematic* equations to another over time. This situation has well-defined control strategies, as discussed next.

1) Stabilization of Manipulation Using Arrays of Actuators: Consider a desired equilibrium point on an alternately orthogonal array. It has contact actuators located at $(2i + 1, 2j + 1)$ with $i, j \in \mathbb{N}$. Their angles are alternately $\frac{\pi}{2}$ and $-\frac{\pi}{2}$. We will denote the velocities of these actuators by $\overline{u}_{(2i+1,2j+1)}$ and the applied force by $u_{(2i+1,2j+1)}$. The system is $\left((\mathcal{U}, \mathcal{D}), \overline{\mathcal{U}}\right)$-reducible by Prop. 7.1, so long as the contact interfaces are *dissipative* when slipping is occurring (i.e., the reaction force is nonzero and in the opposite direction of the slipping). Additionally, all the nontrivial, non-overconstrained kinematics when the center of mass is near $(x, y) = (0, 0)$ are of one of the four forms in Table VII-.1 [13], [12].

For each of the four models in Table VII-.1 a control law is calculated from the Lyapunov function $k(x^2 + y^2 + \theta^2)$ (where k is some constant to be chosen during implementation) by solving $\dot{V} = -V$ for $\overline{u}_{(2i+1,2j+1)}$ subject to the constraint that actuators with the same orientation have the same velocity command $\overline{u}$. Hence, there are two unique inputs $\overline{u}_{(1,1)}$ and $\overline{u}_{(-1,1)}$ in the kinematic description, and including more does not help [13], [12]. Moreover, by virtue of the design methodology, there is a common Lyapunov function. This was shown to provide global stabilization to $(0, 0, 0)$ for the kinematic system in [13], [12].

Figure 6 shows three simulations of an actuator array near a desired equilibrium. For each simulation, going from left to right, the XY location of a manipulated object is shown, the orientation θ, the evolution of σ, and the response of the actuator at $(1, 1)$ in the dynamic simulation as it tracks $\overline{u}_{(1,1)}$. The four actuators near the equilibrium dominate the motion, and the rest are kinematically constrained to match the speeds of $u_{(1,1)}$ and $u_{(-1,1)}$. We use $\mathbb{E}$ from Eq.(11) to estimate σ and Eq. (9) to implement the commands $\overline{u}_{i,j}$ (i.e., the mapping $\pi : \mathcal{I}_{der} \to \mathcal{I}$) with a control gain of $K = 10$ for three different friction laws–Coulomb friction, viscous friction, and stiction, as in Section IV-A. All the responses have an initial condition of $(x_0, y_0, \theta_0) = (0.5, 2, \frac{\pi}{2})$ and the

TABLE I

ORTHOGONAL ACTUATOR ARRAYS (LIKE THOSE SEEN IN FIG. 5 AND FIG. 5) HAVE ALL KINEMATIC STATES, MANY OF WHICH ARE REDUNDANT. THIS FIGURE SHOWS THE FOUR DISTINCT EQUATIONS OF MOTION THAT CAN OCCUR. NOTE THAT SO LONG AS $\overline{u}_{(1,1)}$ $(= \overline{u}_{(-1,-1)})$ AND $\overline{u}_{(-1,1)}$ $(= \overline{u}_{(1,-1)})$ ARE NONZERO, THE FOUR STATES CAN BE DISTINGUISHED FROM STATE OUTPUT. IN FACT, JUST OBSERVATION OF θ IS SUFFICIENT FOR DISTINGUISHING THE STATES. MOREOVER, THIS SYSTEM CAN BE STABILIZED TO THE ORIGIN USING THE CONTROL LAW SHOWN AND AN ESTIMATE OF σ [12].

σ	Equations of Motion	Control Law
$\sigma = 1$	$\dot{q} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \overline{u}_{(1,1)} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \overline{u}_{(-1,1)}$	$\overline{u}_{(1,1)} = \frac{-k\theta\,(\theta+x-y)+k\left(\theta^2+x^2+y^2\right)}{x+y}$ $\overline{u}_{(-1,1)} = -k\theta$
$\sigma = 2$	$\dot{q} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \overline{u}_{(1,1)} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \overline{u}_{(-1,1)}$	$\overline{u}_{(1,1)} = k\theta$ $\overline{u}_{(-1,1)} = \frac{k\theta\,(\theta+x+y)-k\left(\theta^2+x^2+y^2\right)}{x-y}$
$\sigma = 3$	$\dot{q} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \overline{u}_{(1,1)} + \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \overline{u}_{(-1,1)}$	$\overline{u}_{(1,1)} = \frac{k\theta\,(\theta-x+y)+k\left(\theta^2+x^2+y^2\right)}{x+y}$ $\overline{u}_{(-1,1)} = k\theta$
$\sigma = 4$	$\dot{q} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \overline{u}_{(1,1)} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \overline{u}_{(-1,1)}$	$\overline{u}_{(1,1)} = -k\theta$ $\overline{u}_{(-1,1)} = \frac{-k\theta\,(-\theta+x+y)-k\left(\theta^2+x^2+y^2\right)}{x-y}$

goal is to stabilize to the origin $(0, 0, 0)$. All responses are qualitatively similar despite the differences in friction law in the actual implementation.

The key is that despite changes in the characteristics of friction, the controller computed in $\mathcal{I}_{der}$ (which only needs to estimate σ) performs well (and similarly to the macro-scale experimental work in [13]). This is because all possible σ yield kinematic equations of motion that can be stably implemented using Eq. (9).

Lastly, note that σ does not change very quickly in this setting. Moreover, looking at the kinematic equations of motion, we see that σ can be estimated based on θ measurements alone (so long as $\overline{u}_1 \neq \overline{u}_2$ and $\overline{u}_1, \overline{u}_2 \neq 0$). In this case sensing θ at 10 Hz would be more than sufficient for purposes of capturing the σ changes. So if d in Eq. (10) is large enough, we can estimate σ using Eq. (11). In comparison to directly identifying $\tau \in \mathcal{L}$, which often requires sampling rates at 1 KHz or more, this is clearly superior from a sensing perspective.

VIII. CONCLUSIONS

This paper considers the use of derived information spaces that arise from the canonical distinction between slipping frictional forces and nonslipping frictional forces. Geometric kinematic reductions play a central role in why this choice is effective in generating useful descriptions of a system, even when a system experiences stick/slip phenomena (which are typically thought of as being dynamic). Both planning and stabilization can be computed in the derived information space, and then implemented in the underlying dynamic space through the use of a stable plan, typically just a backstepping controller in the context of the work presented here. These techniques are illustrated on an two example simulation–an actuator array. Lastly, the derived information space has more limited sensing requirements, both in terms of spatial resolution and temporal resolution.

One of the most pressing areas of future work is understanding the effect of $\pi^{-1} : \mathcal{I} \rightarrow \mathcal{I}_{der}$ on noise and other uncertainties that are naturally represented in $\mathcal{I}$. This is because noisy observations in $\mathcal{I}$ potentially add uncertainty to the estimate in $\mathcal{I}_{der}$, but they do not have to. How to treat this analytically is the subject of ongoing research. Lastly, experimental versions of the example in Section VII is under development in the author's laboratory.

REFERENCES

[1] T. Akiyama and K. Shonu. Controlled stepwise motion in polysilicon micro-structures. *Journal of Microelectromechanical Systems*, 2(3):106–110, 1993.

[2] A. Bicchi and V. Kumar. Robotic grasping and contact: a review. In *IEEE Int. Conf. on Robotics and Automation (ICRA)*, pages 348–353, 2000.

[3] F. Bullo and A. Lewis. *Geometric Control of Mechanical Systems*. Number 49 in Texts in Applied Mathematics. Springer-Verlag, 2004.

[4] F. Bullo and A. Lewis. Low-order controllability and kinematic reductions for affine connection control systems. *SIAM Journal on Control and Optimization*, 44(3):885–908, 2005.

[5] F. Bullo and K. Lynch. Kinematic controllability and decoupled trajectory planning for underactuated mechanical systems. In *IEEE Int. Conf. on Robotics and Automation (ICRA)*, pages 3300–3307, 2001.

[6] F. Bullo and K. M. Lynch. Kinematic controllability for decoupled trajectory planning in underactuated mechanical systems. *IEEE Transactions on Robotics and Automation*, 17(4):402–412, August 2001.

[7] A. Filippov. *Differential Equations with Discontinuous Right-Hand Sides*. Kluwer, 1988.

[8] S. LaValle. *Planning Algorithms*. Cambridge Press, 2006.

[9] A. Lewis. When is a mechanical control system kinematic? In *Proc.* 38^{th} IEEE *Conf. on Decision and Control*, pages 1162–1167, Dec. 1999.

[10] R. Linderman and V. Bright. Nanometer precision positioning robots utilizing optimized scratch drive actuators. *Sensors and Actuators A*, 91:292–300, 2001.

[11] J. Luntz, W. Messner, and H. Choset. Distributed manipulation using discrete actuator arrays. *Int. J. Robotics Research*, 20(7):553–583, July 2001.

[12] T. D. Murphey. On multiple model control for multiple contact systems. *Automatica*, Accepted for Publication.

[13] T. D. Murphey and J. W. Burdick. Feedback control for distributed manipulation with changing contacts. *International Journal of Robotics Research*, 23(7/8):763–782, July 2004.

[14] T. D. Murphey and J. W. Burdick. The power dissipation method and kinematic reducibility of multiple model robotic systems. *IEEE Transactions on Robotics*, 22(4):694–710, August 2006.

[15] H. Olsson, K. Astrom, C. C. de Wit, M. Gafvert, and P. Lischinsky. Friction models and friction compensation. *European Journal of Control*, 4(3):176–195, 1998.

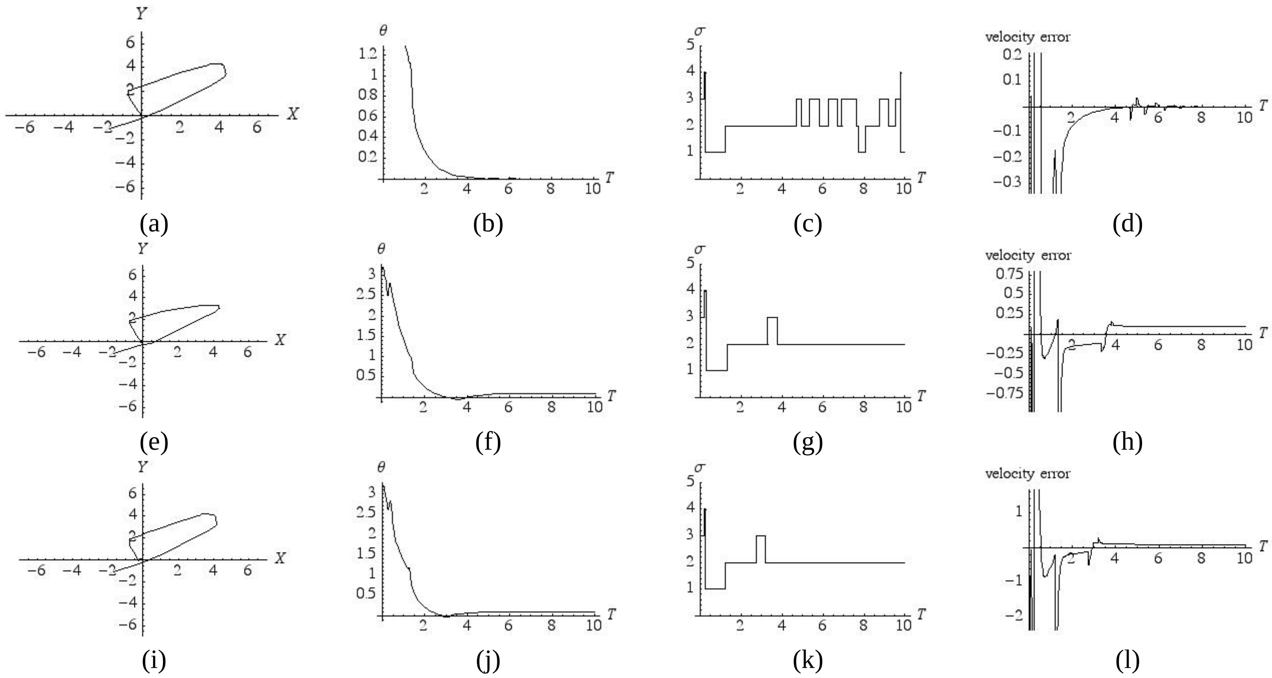

Fig. 6. Three simulations with different choices of friction model. From left to right, the plots are the XY trajectory of an object supported by an actuator array, the orientation θ as a function of time, the kinematic state σ as a function of time, and lastly the response of the actuator at $(1, 1)$ as it tracks the desired velocity $\overline{u}_{(1,1)}$. The simulations are for viscous friction (a-d), coulomb friction (e-h), and stiction friction (i-l). Parameters used were $m = 1$, $J = 5$, $\mu_S = 1.1$, $\mu_K = 1$, and $K = 10$ in Eq. (9).

[16] G. Pappas, G. Laffierier, and S. Sastry. Hierarchically consistent control sytems. *IEEE Trans. Automatic Control*, 45(6):1144–1160, June 2000.

[17] M. Vidyasagar. *Nonlinear Systems Analysis*. Prentice Hall, 1978.

APPENDIX

In discussing kinematic reductions, we follow [3], [4]. A *simple mechanical control system* consists of a manifold Q of dimension n, a Riemannian metric G that defines the kinetic energy, a set of constraints represented as a constraint distribution D, and a set of external forces. Associated with a Riemannian metric G are what are called *Christoffel symbols.*

Definition 1.1: The *Christoffel symbols* associated with the metric G are

$$ {}^G\Gamma^i_{jk} = \frac{1}{2} G^{il} \left(\frac{\partial G_{jl}}{\partial q^k} + \frac{\partial G_{kl}}{\partial q^j} - \frac{\partial G_{jk}}{\partial q^l} \right) \qquad (12) $$

where summation over repeated indices is implied used unless otherwise stated, and upper indices indicate the inverse. Also associated with the Riemannian metric is the *affine connection*, which assigns to a pair of vector fields X and Y another vector field $\nabla_X Y$. This is referred to as the *covariant derivative* of Y with respect to X.

Definition 1.2: In coordinates, the covariant derivative of Y with respect to X is

$$ {}^G\nabla_X Y = \left(\frac{\partial Y^i}{\partial q^j} X^j + \Gamma^i_{jk} X^j Y^k \right) \frac{\partial}{\partial q^j} \qquad (13) $$

With this, the Euler-Lagrange equations can be written as

$$ {}^G\nabla_{c'(t)} c'(t) = u^a(t) Y_a(c(t)) \qquad (14) $$

where $t \mapsto c(t)$ is a path on Q and $c'(t) = \frac{d}{dt}c(t)$ [3], [4]. In this equation ${}^G\nabla$ is the constrained affine connection associated with the Riemannian metric (kinetic energy) G and Y_a are force vector fields associated with forces u^a. In coordinates this is written as:

$$ \ddot{q}^i + {}^G\Gamma^i_{jk} \dot{q}^j \dot{q}^k = u^\alpha \; Y^i_\alpha. \qquad (15) $$

Constrained systems, those control systems whose trajectories must lie in some distribution D, can also be described by Eq. (14). However, the affine connection must be modified in order to incorporate the constraints. Let D be a distribution on Q and let $D^\perp$ denote the G-orthogonal complement of D. Moreover, let $P : TQ \to TQ$ be a G-orthogonal projection operator onto D and let $P' : TQ \to TQ$ be a G-orthogonal projection onto $D^\perp$. Lastly, let $A(q)$ be any invertible $(1, 1)$ tensor field on Q and let $B(q)$ be its inverse. Then, the Euler-Lagrange equations can be written as Eq. (15) where the Chrisoffel symbols are:

$$ {}^A\Gamma^i_{jk} = {}^G\Gamma^i_{jk} + B^i_l \frac{\partial (AP')^l_j}{\partial q^k} + B^{i\,G}_l \Gamma^l_{km} (AP')^m_j - B^{i\,G}_l \Gamma^m_{kj} (AP')^l_m $$

where, again, $A(q)$ is *any* invertible $(1, 1)$ tensor on Q. In order to add forces, we must ensure the forces comply with the constraints. Hence, the final equations of motion are:

$$ {}^G\nabla_{c'(t)} c'(t) = u^a(t) P_j Y^j_a(c(t)) \qquad (16) $$

or in coordinates:

$$ \ddot{q}^i + {}^A\Gamma^i_{jk} \dot{q}^j \dot{q}^k = u^\alpha \; P_j Y^i_\alpha. \qquad (17) $$

Therefore, simple mechanical control systems *all* can be represented using an affine connection. For more details and examples worked out in detail, refer to Bullo and Lewis [3].