Multiple Instantaneous Collisions in a Variational Framework

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***Abstract*— Variational integrators have become of interest to the controls community, particularly since in recent years they have been used in the context of optimal control. This paper discusses how to resolve simultaneous impacts in a variational context, assuming that all boundaries of the impacting bodies are differentiable. We demonstrate how this analysis works for arbitrary numbers of simultaneous impacts and derive discrete time algorithms for resolving these impacts numerically. We illustrate our results, both in the continuous and discrete domain, using Newton's cradle as an example.**

I. Introduction

Optimal control using variational integrators [9], [4], [10]–typically referenced as Discrete Mechanics Optimal Control (DMOC)–has proven to be a particularly useful numerical approach for control of walking [11] and other systems that exhibit highly nonlinear behavior with nonsmooth impacts. Hence, understanding simultaneous impacts–impacts that literally happen at the same instant–is important for numerical implementations of DMOC techniques. An example of such a system is billiard balls that are packed together before a break. Much work has been done in this direction, the most common way of dealing with such a system being linear complementarity problem (LCP) formulations [12], [1], [2]. While such approaches guarantee existence of solutions, they do not provide much toward uniqueness, and due to the fact that all potential impacts in one time step are resolved simultaneously, regardless of their nature, it is not clear how close such solutions are to physical reality. Other methods include adding penalty potentials to contact surfaces [6], impulse-space based approaches [3] and methods centered in nonsmooth analysis [7], [10], [5]. While all these methods have strong points, they also present major drawbacks, leaving the question of a general approach dealing with collisions, and especially multiple simultaneous collisions, open.

We present a variational approach to posing and solving the problem of collision. This approach has the benefit of avoiding any impulsive or nonsmooth analysis, and stems directly from first principles. We extend this method to dealing with multiple simultaneous collisions, and apply it, as an example, to Newton's cradle. In the case of this system our results are unique, and match the real world evolution the system.

The paper is organized as follows:

Section II develops a mathematical formalism for dealing with simultaneous impacts in a continuous time setting. In section II-A we use, as an example, a three ball Newton's cradle, a system in which conservation of energy and conservation of momentum predict a continuum of valid solutions in the case of elastic impacts, contrary to the unique outcome observed in practice. We show that applying a variational principle to this problem leads to a unique and expected outcome in the continuous time setting.

In section III we derive discrete versions of the equations and describe how one would build variational integrators that can detect and resolve impacts, multiple impacts in a time step and, most importantly, multiple simultaneous impacts. These equations can be solved with standard root solving techniques, provided one has access to multiple partial derivatives of the continuous time Lagrangian. A way of describing and simulating systems which gives access to these derivatives can be found in [8] and related work.

Section IV presents the results from simulating a three ball Newton's cradle using the methods described in previous sections and comment on a change of coordinates which is needed in order to avoid numerical error.

Conclusions and future work can be found in section V.

II. Continuous Time

Consider the case of n identical frictionless rigid spheres arranged collinearly. The system can be easily represented in generalized coordinates by the x coordinate of each sphere so that $q = [x_1, x_2, ..., x_n]^{\mathrm{T}}$. The Lagrangian is

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^{\mathrm{T}} M \dot{q},$$

and only consists of kinetic energy, since there is no potential energy in this problem. The mass matrix is

$$M = mI_{n\times n}.$$

The boundaries between the spheres can be described using real valued functions $\phi_i : Q \to \mathbb{R}$ such that

$$\phi_i(q) > 0, \quad q \in C_i, \tag{1a}$$

$$\phi_i(q) < 0, \quad q \in Q/C_i, \tag{1b}$$

$$\phi_i(q) = 0, \quad q \in \partial C_i, \tag{1c}$$

where C_i is the admissable set with respect to boundary i and ∂C_i describes the boundary of the admissable set. The condition that $q \in \partial C_i$ thus becomes $\phi_i(q) = 0$.

Note that in the described case of n spheres, boundary ∂C_i describes the boundary between spheres i and $i+1$. Thus, for n spheres, i goes up to $n-1$. Also, in our example, the most straightforward boundary functions can be chosen to be

$$\phi_i(q) = (x_{k+1} - x_i) - 2r, \quad \forall k \in \{1, 2, ..., (n-1)\},$$

although any choice that respects (1c) would work equally well.

Assume there is a collision across boundary ∂C_i at time t_i. Both the configuration q and its derivative $\dot{q}$ have to be continuous at every time other than at the collision time, when we require q to be continuous, but not necessarily smooth: we both allow and expect a discontinuity in $\dot{q}$ at time $t = t_i$. As such, the velocity before impact will generally be different from the velocity after impact: $\dot{q}(t_i^-) \neq \dot{q}(t_i^+)$. In fact, when considering the continuous problem, the requirement that the configurations right after the impact lie in C_i becomes a requirement on the exit velocity

$$\nabla\phi_i(q(t_i)) \cdot \dot{q}(t_i^+) \geq 0, \tag{2}$$

which states that the system has to be moving away from the impact surface right after the impact.

We calculate the variation of the action, with respect to variations in both the curve $q(t)$ and the impact time t_i:

$$\begin{aligned}
&\delta \int_0^{\mathrm{T}} L(q(t), \dot{q}(t))\, dt \\
&\quad = \delta \int_0^{t_i} L(q(t), \dot{q}(t))\, dt + \int_{t_i}^{\mathrm{T}} L(q(t), \dot{q}(t))\, dt \\
&\quad = \int_0^{t_i^-} \left[\frac{\partial L}{\partial q} \cdot \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q}\right] dt + L(q, \dot{q}) \cdot \delta t_i \Big|_{t_i^-} \\
&\qquad + \int_{t_i^+}^{\mathrm{T}} \left[\frac{\partial L}{\partial q} \cdot \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q}\right] dt - L(q, \dot{q}) \cdot \delta t_i \Big|_{t_i^+} \\
&\quad = \int_0^{t_i^-} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] \cdot \delta q dt \\
&\qquad + \int_{t_i^+}^{\mathrm{T}} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] \cdot \delta q dt \\
&\qquad - \left[\frac{\partial L}{\partial \dot{q}} \cdot \delta q + L \cdot \delta t_i\right]_{t_i^-}^{t_i^+},
\end{aligned}$$

where we used the Leibniz rule, integration by parts and the condition that $\delta q(T) = \delta q(0) = 0$. Requiring that the variation in the action be zero for all δq away from t_i gives the Euler-Lagrange equations:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0.$$

The jump term must also be zero for all possible variations at the time of impact. To describe the space of all possible variations we start by differentiating $\phi_i(q(t_i)) = 0$ with respect to variations in $q(t_i)$ and in t_i:

$$\begin{aligned}
&D\phi_i(q(t_i)) \cdot [\delta q(t_i) + \dot{q}(t_i)\, \delta t_i] = 0, \text{ or} \\
&\nabla\phi_i(q(t_i))^{\mathrm{T}} [\delta q(t_i) + \dot{q}(t_i)\, \delta t_i] = 0.
\end{aligned}$$

We look for a basis set that spans the space of allowable pairs $(\delta t_i, \delta q(t_i))$ as per the conditions above. First, let us set $\delta t_i = 1$. This gives us that the pair $(1, -\dot{q}(t_i))$ satisfies our conditions, and hence we will use it as one of the bases. Next, let us take $\delta t_i = 0$. The allowable pairs under this assumption become all pairs $(0, \delta q(t_i))$ such that

$$\nabla\phi_i(q(t_i))^{\mathrm{T}} \delta q(t_i) = 0, \quad \delta q(t_i) \neq 0.$$

These pairs, along with $(1, -\dot{q}(t_i))$, form a linearly independent set that spans an n dimensional space, making them a basis for the set of all allowable variations. Plugging these pairs into the jump term gives us:

$$\begin{aligned}
\frac{\partial L}{\partial \dot{q}}\Big|_{t_i^-}^{t_i^+} \cdot \delta q(t_i) = 0, \\
\left[\frac{\partial L}{\partial \dot{q}} \cdot \dot{q} - L\right]_{t_i^-}^{t_i^+} = 0,
\end{aligned}$$

where $\nabla\phi_i(q(t_i))^{\mathrm{T}} \delta q(t_i) = 0$ and $\delta q(t_i) \neq 0$. The former equation is a vector equation of dimension n and gives the conservation momentum during the impact across the boundary ∂C_i. The latter is a scalar equation, representing the conservation of energy during the same impact.

Since all impacts are the same physical phenomena and obey the same laws and principles, the equations above apply to impacts at any time t_i across any surface ∂C_i. Assume we have m collisions, occuring at times $t_{i+1} \geq t_i$. For example, in the case of the system of n colinear spheres (Newton's cradle) with $q(t_1^-) = [v, 0, ..., 0]^{\mathrm{T}}$, we will have that $m = n-1$. Now, let us take the limit as $t_{i+1} \to t_i$, that is assume the collisions are all instantaneous. Due to the fact that $\dot{q}$ has to be continuous at all non-collision times, we have that, as the intervals $[t_i, t_{i+1}]$ shrink the values of $\dot{q}$ at both ends of the interval become identical, $\dot{q}(t_{i+1}^-) \to \dot{q}(t_i^+)$. Adding this to our previous set of equations, and writing the conservation of momentum using Lagrange multipliers, gives us

$$\frac{\partial L}{\partial \dot{q}}\Big|_{t_i^-}^{t_i^+} + \lambda_i \nabla\phi_i(q(t_i)) = 0, \quad \forall i \in \{1, ..., m\} \tag{4a}$$

$$\left[\frac{\partial L}{\partial \dot{q}} \cdot \dot{q} - L\right]_{t_i^-}^{t_i^+} = 0, \quad \forall i \in \{1, ..., m\} \tag{4b}$$

$$\dot{q}(t_{i+1}^-) - \dot{q}(t_i^+) = 0, \quad \forall i \in \{1, ..., m-1\}, \tag{4c}$$

which is a system of $(2n+1)m - n$ equations to be solved for $(2n+1)m - n$ variables, namely $\dot{q}(t_i^+)$, $\dot{q}(t_i^+)$ and λ_i, excluding $\dot{q}(t_1^-)$, which is known.

Eq. (4c) is written in a form which brings forth the physical nature of the system but somewhat obscures its mathematical topology. Some algebraic manipulation gives us the equivalent system

$$\frac{\partial L}{\partial \dot{q}}\Big|_{t_i^-}^{t_i^+} + \sum_{i=1}^{k} \lambda_i \nabla\phi_i(q(t_i)) = 0, \quad \forall i \in \{1, ..., m\} \tag{5a}$$

$$\left[\frac{\partial L}{\partial \dot{q}} \cdot \dot{q} - L\right]_{t_i^-}^{t_i^+} = 0, \quad \forall i \in \{1, ..., m\}, \tag{5b}$$

to be solved for $\dot{q}(t_i^+)$ and λ_i for all i. First, notice that for every i, choosing $\lambda_i = 0$ gives a trivial solution in which the system is unchanged after the collision. This solution includes no information regarding the impact surface, and thus corresponds to the bodies freely passing through each other. Moreover, the exit velocity $\dot{q}(t_i^+)$, being equal to the velocity before impact, $\dot{q}(t_i^-)$, will undoubtably violate the constraint given in (2). In all subsequent calculations, including those covering the discrete time case, we will implicitly exclude this sort of trivial solution in which the Lagrange multipliers are zero, and assume that the condition $\lambda \neq 0$ has to hold.

In deriving (5b) we assumed a certain order of impacts before taking the limit. Although in the case of Newton's cradle this ordering is clear, it may not be so in other cases, and the reader might be wondering just how general our results are. Also, the questions arises whether a different assumed order of impacts would have led to a different result. Current work suggests that the order in which collisions are resolved is unimportant in some systems (like Newton's cradle) but important in others (certain billiard ball configurations). Thus, regardless of the assumed order of impacts, our method will always return the same result for Newton's cradle. However, this is not true of all possible systems, although the number of outcomes is finite. We are currently investigating whether an approach similar to the projection discussed in [4], [7], and [10].

Note that the order in which collisions happen has nothing to do with any ordering in time, but with the way in which the resolution of one boundary affects the rest of the equations, since all collisions happen at the exact same moment in time. The word order, as used here, does not refer to a timing order but rather to a topological order, a relation that depends upon the static geometry of the system rather than its time evolution.

A. *Linear Newton's Cradle in Continuous Time*

To illustrate, let's consider Newton's cradle with three spheres arranged in a line and $\dot{q}(t_1^-) = [v, 0, 0]^{\mathrm{T}}$ for some q in which all three spheres are touching. This is simply saying that two of them were at rest, while the first one is hitting them with a velocity v. In this case the boundary normals are

$$\nabla\phi_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \nabla\phi_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

The equations for this particular case look like

$$\begin{aligned} m\left[\dot{q}(t_1^+) - \dot{q}(t_1^-)\right] + \lambda_1 \nabla\phi_1 = 0, \\ m\left[\dot{q}(t_2^+) - \dot{q}(t_2^-)\right] + \lambda_2 \nabla\phi_2 = 0, \\ \dot{q}(t_1^+)^{\mathrm{T}}\,\dot{q}(t_1^+) - \dot{q}(t_1^-)^{\mathrm{T}}\,\dot{q}(t_1^-) = 0, \\ \dot{q}(t_2^+)^{\mathrm{T}}\,\dot{q}(t_2^+) - \dot{q}(t_2^-)^{\mathrm{T}}\,\dot{q}(t_2^-) = 0, \\ \dot{q}(t_2^-) - \dot{q}(t_1^+) = 0. \end{aligned}$$

To get the full system in all scalar variables we substitute our previously shown values, use the last equation to eliminate $\dot{q}(t_2^-)$ and name $\dot{q}(t_1^+) = [v_{11}, v_{12}, v_{13}]^{\mathrm{T}}$ and $\dot{q}(t_2^+) = [v_{21}, v_{22}, v_{23}]^{\mathrm{T}}$. This gives us

$$\begin{aligned} &\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = -\frac{\lambda_1}{m}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix}, \\ &\begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = -\frac{\lambda_2}{m}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix}, \\ &v_{11}^2 + v_{12}^2 + v_{13}^2 = v^2, \\ &v_{21}^2 + v_{22}^2 + v_{23}^2 = v_{11}^2 + v_{12}^2 + v_{13}^2. \end{aligned}$$

This system is easily solvable using basic algebra and has unique solutions if we discard those in which the Lagrange multipliers are zero (i.e. $\lambda_i \neq 0$). The solution is, as expected

$$\dot{q}(t_1^+) = \begin{bmatrix} 0 \\ v \\ 0 \end{bmatrix}, \quad \dot{q}(t_2^+) = \begin{bmatrix} 0 \\ 0 \\ v \end{bmatrix}.$$

Using the same steps, one finds that if the initial condition is instead given as $\dot{q}(t_1^-) = [v, v, 0]^{\mathrm{T}}$, then the final solution is $\dot{q}(t_2^+) = [0, v, v]^{\mathrm{T}}$. In other words, if two spheres are initially displaced, two spheres will split off at the end. This is also consistent with the real behavior of Newton's cradle.

III. Discrete Time

To start with, let us consider a sequence of the form $(t_0, q_0), (t_1, q_1), ..., (t_n, q_n)$, where $q_k = q(t_k)$. For simplicity, consider a fixed time step, that is $h = t_{k+1} - t_k$ for all k. Now we define a *discrete* Lagrangian that approximates the action integral over one time step:

$$L_d(q_k, q_{k+1}, h) = L\left(\bar{q}, \bar{\bar{q}}\right) \approx \int_{t_k}^{t_{k+1}} L(q(\tau), \dot{q}(\tau))\, d\tau,$$

where we used the midpoint rule $\bar{q} = (q_{k+1} + q_k)/2$ and $\bar{\bar{q}} = (q_{k+1} - q_k)/h$. This leads to approximating the action integral with an action sum

$$S = \sum_{k=0}^{n-1} L_d(q_k, q_{k+1}, h). \tag{8}$$

Minimizing (8) gives us the discrete Euler-Lagrange equation:

$$D_2 L_d(q_{k-1}, q_k, h) + D_1 L_d(q_k, q_{k+1}, h) = 0. \tag{9}$$

This equation uses the previous two states to find the next state, thus defining a mapping of the form

$$(q_{k-1}, q_k) \to q_{k+1}.$$

A. *Implementation Note*

In order to solve (9) and all other similar equations in this work, we assume we always have access to the system's *continuous* Lagrangian and several of its partial derivatives.

Thus, using the notation just described, we would make the substitutions

$$D_1 L_d(q_k, q_{k+1}, h) = \frac{h}{2}\partial_q L\left(\bar{q}, \bar{\dot{q}}\right) - \partial_{\dot{q}} L\left(\bar{q}, \bar{\dot{q}}\right),$$
$$D_2 L_d(q_k, q_{k+1}, h) = \frac{h}{2}\partial_q L\left(\bar{q}, \bar{\dot{q}}\right) + \partial_{\dot{q}} L\left(\bar{q}, \bar{\dot{q}}\right),$$
$$D_3 L_d(q_k, q_{k+1}, h) = L\left(\bar{q}, \bar{\dot{q}}\right) - \bar{\dot{q}}\partial_{\dot{q}} L\left(\bar{q}, \bar{\dot{q}}\right)$$

wherever they are applicable. We also assume we have access to all second partial derivatives of the Lagrangian, which is needed for accurately implementing a root search algorithm for equations such as (9). While for simple systems calculating the derivatives of the Lagrangian might not be a problem, it certainly becomes very complicated for large systems involving multiple types of joints. As this is not the focus of this work, we point the reader to [8] for further details on how one could gain easy access to these needed functions.

B. One Collision

Now, assume that we have determined that an impact happens during the kth time step, more precisely between t_{k-1} and t_k. This can be determined using a collision detection algorithm, a simple implementation of which would be to check for negative values of $\phi_i(q(t_k))$ at each time step. Such values would mean that the corresponding interpenetration condition would be violated by the value of t_k given by (9) and as such we need to find it from equations that assume one or more impacts have occured. Now, since we are interested mainly in what happens at the collision and not outside it, let us refer to t_{k-2} as t_o (for t_{old}), t_{k-1} as t_c (for $t_{current}$), and t_k as t_n (for t_{new}). Let the collision time be $t_1 = t_c + \alpha_1 h$, with $\alpha_1 \in [0, 1]$. We will denote the value of the configuration at the time of impact t_1 as q_1. Finally, let the set of acceptable configurations be C_1 and its boundary be ∂C_1. Fig. 1 is a sketch illustrating both the algorithm and the notation used around a point of impact.

Applying the same variational principles as before over the interval $[t_{k-1}, t_k]$, we get the following set of equations:

$$D_2 L_d(q_o, q_c, h) + D_1 L_d(q_c, q_1, \alpha_1 h) = 0, \quad \text{(11a)}$$
$$\phi_1(q_1) = 0, \quad \text{(11b)}$$
$$D_3 L_d(q_c, q_1, \alpha_1 h) - D_3 L_d(q_1, q_n, (1-\alpha_1)h) = 0, \quad \text{(11c)}$$
$$\lambda_1 \nabla\phi_1(q_1) + D_2 L_d(q_c, q_1, \alpha_1 h) \quad \text{(11d)}$$
$$+ D_1 L_d(q_1, q_n, (1-\alpha_1)h) = 0, \quad \text{(11e)}$$

where the unknowns are q_1, α_1 and q_n. It might be somewhat enlightening to note that (11c) and (11d) can be readily interpreted as discrete versions of conservation of energy and momentum respectively. Eq. (11b) states that q_i must lie on the boundary and (11a) is a variable step size version of (9). The last equation affected by the impact is

$$D_2 L_d(q_1, q_n, (1-\alpha_1)h) + D_1 L_d(q_n, q_{k+1}, h) = 0, \quad (12)$$

which is a variable step size version of (9) that mirrors (11a).

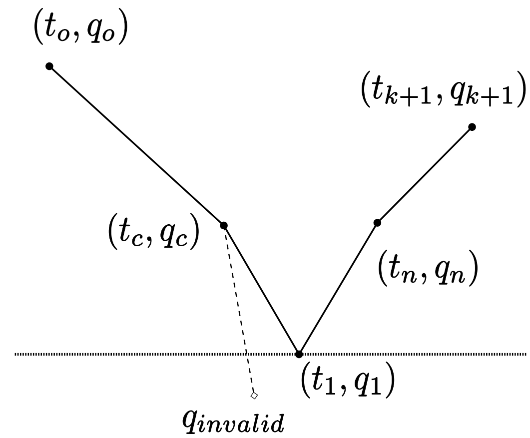

Fig. 1. Sketch illustrating the algorithm and notation used to deal with a simple impact. Here q_o and q_c are the two consecutive configurations before the impact, $q_{invalid}$ is the new configuration if there were no impact, t_1 is the time of impact, q_1 is the configuration at impact and q_n is the correct post-impact configuration that replaces $q_{invalid}$. After the impact, the simulation continues with the new point at q_{k+1}

C. Multiple Disctict Collisions

It is possible that the time between two consecutive impacts is small enough for both of the impact times to fall inside the same time step. Fig. 2 sketches this situation for two distinct impacts across two distinct boundaries. Algorithmically we will know this has happened if the

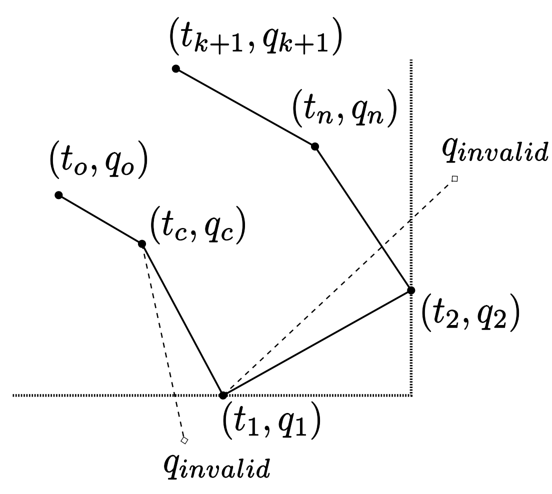

Fig. 2. Sketch illustrating the algorithm used to dealing with two consecutive impacts in the time step between $t_{k-1} = t_c$ and $t_k = t_n$. In this case, solving the first collision fails to give a valid result with respect to a second surface. This indicates a collision with that surface, which is found to occur at time t_2 and configuration q_2. The result of this collision is valid and integration continues from q_n to q_{k+1}

q_n returned by (11) does not satisfy the interpenetration constraints. If this is the case, we need to solve a slightly different system of equations to find q_n. Assume m collisions in the same time step, each at time $t_i = t_c + \sum_{j=1}^{i} \alpha_j h$ and across boundaries C_i, where $\alpha_i \in (0,1)$. Then (11) becomes

$$\begin{aligned}
D_2 L_d(q_o, q_c, h) + D_1 L_d(q_c, q_1, \alpha_1 h) &= 0, \\
\phi_1(q_1) &= 0, \\
D_3 L_d(q_c, q_1, \alpha_1 h) - D_3 L_d(q_1, q_2, \alpha_2 h) &= 0, \\
\lambda_1 \nabla \phi_1(q_1) + D_2 L_d(q_c, q_1, \alpha_1 h) + D_1 L_d(q_1, q_2, \alpha_2 h) &= 0, \\
\phi_2(q_2) &= 0, \\
&\cdots \\
D_3 L_d(q_{m-1}, q_m, \alpha_m h) \\
-D_3 L_d\left(q_m, q_n, \left(1 - \sum_{i=1}^{m} \alpha_i\right) h\right) &= 0, \\
\lambda_m \nabla \phi_m(q_m) + D_2 L_d(q_{m-1}, q_m, \alpha_m h) \\
+D_1 L_d\left(q_m, q_n, \left(1 - \sum_{i=1}^{m} \alpha_i\right) h\right) &= 0,
\end{aligned} \tag{13}$$

which are to be solved for all q_i, α_i and ultimately q_n. Next, we solve the analogous of (12)

$$D_2 L_d\left(q_m, q_n, \left(1 - \sum_{i=1}^{m} \alpha_i\right) h\right) + D_1 L_d(q_n, q_{k+1}, h) = 0,$$

and continue with solving (9) in order to get the evolution away from the point of impact.

D. Multiple Instantaneous Collisions

Before handling instantaneous collisions algorithmically in the discrete time setting, we need to specify a way to detect that such a collision occured. In the continuous case we assumed that we know this is the case, which is not enough for a working simulation. It is useful to note that, while multiple collisions in the same time step can occur across the same surface of impact ∂C, this would not be possible in the case of instantaneous collisions: at one time t_i the system can only impact one surface, which we call ∂C_i. In other words, if two consecutive impacts occur across the same surface, the time between them has to be finite, and hence covered by the previous section.

The other algorithmically important observation is that, just as in the continuous time case, if two collisions across two different surfaces happen at the exact same time, we expect the solution for $q_i \in \partial C_1$ to also satisfy $q_i \in \partial C_2$. This is what we will use to determine if a multiple collision, detected as described in the previous section, happens at the exact same time. When a multiple impact is detected to have occured in the same time step, we will first check to see if the impact point lies on more that one surface and solve the impact equations in the limit $\alpha \to 0$.

To illustrate this, consider the case of m impacts covered in the previous section, but now let the jth and $j+1 = k$th impacts happen simultaneously. In other words, we solve for q_j using the condition that $q_j \in \partial C_j$, we attempt to solve for q_n assuming that j was the last impact, but we get an erronous result (showing that there must be extra collisions to solve for), and finally we notice that $q_j \in \partial C_k$ also holds. As a conseqence, we solve for the same equations, but in the limit $\alpha_k \to 0$.

The affected terms and their limits are

$$\begin{aligned}
\lim_{\alpha_k \to 0} \left[D_1 L_d(q_j, q_k, \alpha_k h)\right] &= -\frac{\partial L}{\partial \dot{q}}\left(q_j, \dot{q}_j\right), \\
\lim_{\alpha_k \to 0} \left[D_2 L_d(q_j, q_k, \alpha_k h)\right] &= \frac{\partial L}{\partial \dot{q}}\left(q_j, \dot{q}_j\right), \\
\lim_{\alpha_k \to 0} \left[D_3 L_d(q_j, q_k, \alpha_k h)\right] &= L\left(q_j, \dot{q}_j\right) - \frac{\partial L}{\partial \dot{q}}\left(q_j, \dot{q}_j\right) \dot{q}_j.
\end{aligned}$$

We then simply substitute these terms into (13) in the apropriate places and remove the condition $q_k \in \partial C_k$ (since we replaced it with the verified assumption that $q_k = q_j \in \partial C_j \cap \partial C_k$). In its stead we gained a new variable to solve for, namely $\dot{q}_j$.

IV. Numerical Results

Using the mathematical and algorithmic framework developed in the previous sections we modeled and simulated a system comprising of three spheric pendulums, as shown in Fig. 3. Variations of this system can be found in any toy store with the name Newton's cradle and its behavior is easily distinguishable and unique. The initial conditions in our simulation matched the ones described in section II-A, where initially two of the spheres are at rest at the equilibrium point and the third has a slight displacement. At the time of impact we expect the kinetic energy to be transferred to the opposite sphere, while spheres one and two remain motionless.

Fig. 3 shows the time evolution of the spheres' velocities, along with sketched configurations at selected times.

We should note that, like any numerical simulation, ours too is prone to numerical errors, specifically during root finding. As such, the exact position of the spheres after an impact is only approximate. One can imagine that if the exact resting position were to be an irrational number (e.g. $\pi/2$) no amount of numerical approximation could give us a good enough answer, resulting eventually in strange, unwanted behavior. We found that choosing an apropriate coordinate system so that the collision happens around zero, the root finding combined with apropriate truncation gives much better and consistent results. A way of applying this coordinate transformation automatically for every collision is needed and it is our hope that future work will shed more light on this issue.

V. Conclusions and Future Works

In the previous sections we described a variational formalism for dealing with multiple rigid body collisions at the same moment in time and across multiple distinct surfaces of impact. We have found analytically that this formalism predicts a unique solution in at least one case, Newton's crade. We presented discrete equations for several classes of impacts, the algorithm needed to solve them and we presented numerical results in order to illustrate these methods.

Since our methods are based on variational integrators, they are the foundation for building DMOC algorithms

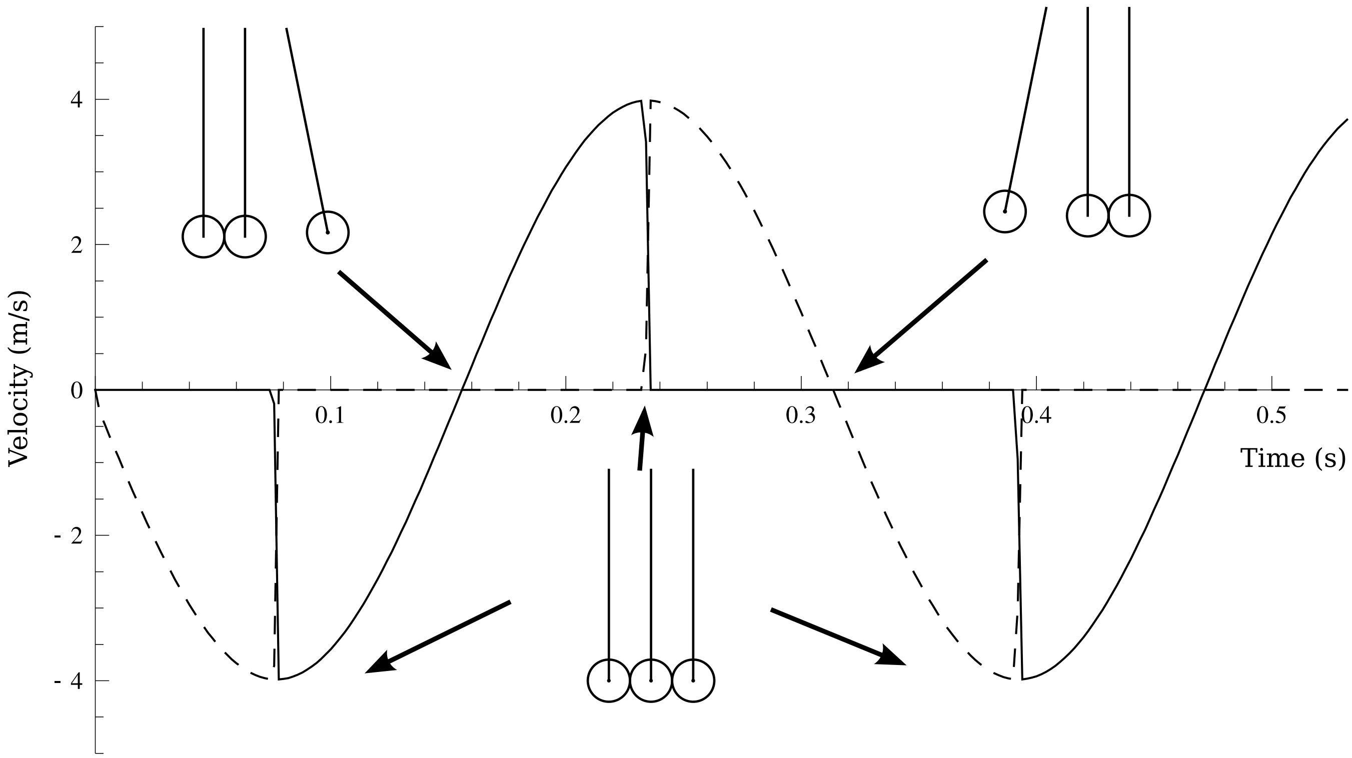

Fig. 3. Numerical results of the simulation implementing a three sphere Newton's Cradle. The continuous line shows the velocity of the right sphere while the dotted line shows the velocity of the left sphere. The sketches show the configuration of the system at the times pointed to by the corresponding arrows.

that deal multiple instantaneous collisions in a natural and principled fashion. Future works will deal with building such DMOC strategies for systems with multiple impacts.

While variational integrators have been shown to preserve integrals of motion, it is not yet entirely clear that our approach in dealing with simultaneous impacts, which adds the continuous time variables $\dot{q}_i$, does the same thing for all possible systems. A deeper mathematical investigation into the properties of the discrete mapping introduced in this work is in order, and will be addressed in future works.

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